

MODULAR SPRINGER CORRESPONDENCE, DECOMPOSITION MATRICES AND BASIC SETS

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ABSTRACT. The Springer correspondence makes a link between the characters of a Weyl group and the geometry of the nilpotent cone of the corresponding semisimple Lie algebra. In this article, we consider a modular version of the theory, and show that the decomposition numbers of a Weyl group are particular cases of decomposition numbers for equivariant perverse sheaves on the nilpotent cone. We give some decomposition numbers which can be obtained geometrically. In the case of the symmetric group, we show that James' row and column removal rule for the symmetric group can be derived from a smooth equivalence between nilpotent singularities proved by Kraft and Procesi. We give the complete structure of the Springer and Grothendieck sheaves in the case of SL_2 . Finally, we determine explicitly the modular Springer correspondence for exceptional types.

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1. INTRODUCTION

1.1. Modular Springer correspondence. Let G be a connected reductive group over a finite field \mathbb{F}_q of characteristic p , with Frobenius endomorphism F . In 1976, Springer defined a correspondence making a link between the irreducible ordinary representations of the Weyl group W of G (over a field of characteristic zero \mathbb{K}) and the nilpotent orbits in the semi-simple Lie algebra \mathfrak{g} of G [Spr76], under some restrictions on p and q . More precisely, he constructed these representations in the top cohomology of some varieties associated to the different nilpotent orbits, the Springer fibers, which are the fibers of Springer's resolution of the nilpotent cone. Each irreducible representation can be associated with a nilpotent orbit and an irreducible G -equivariant local system on this orbit. Springer's original approach was in terms of trigonometric sums, which are (up to a scalar) Fourier transforms of characteristic functions of G^F -orbits on \mathfrak{g}^F . This required to consider varieties over a base field of characteristic p , and ℓ -adic cohomology.

Then many other approaches to Springer correspondence were discovered. For example, Kazhdan and Lusztig found a topological approach [KL80], and Slodowy constructed Springer representations by monodromy [Slo80a]. Links between different constructions were established, as in Hotta's work [Hot81]. In the early 1980's, the emergence of intersection cohomology, discovered by Goresky and MacPherson [GM80, GM83], and the theory of perverse sheaves developed by Beilinson, Bernstein, Deligne and Gabber [BBD82], allowed Lusztig, Borho and MacPherson to reinterpret the correspondence with these new tools [Lus81, BM81]. Later, Hotta and Kashiwara found an approach using a Fourier transform for \mathcal{D} -modules, when the base field is \mathbb{C} [HK84], and Brylinski adapted this idea to the case of a base field of characteristic p , using a Fourier-Deligne transform for perverse sheaves in ℓ -adic cohomology [Bry86].

Let us explain briefly the construction. Let \mathbb{K} be a finite extension of \mathbb{Q}_ℓ , with $\ell \neq p$. We call *ordinary Springer sheaf* the (derived) direct image of the constant perverse sheaf \mathbb{K} (suitably shifted) under the Springer resolution $\pi_{\mathcal{N}}$ of the nilpotent cone. By the Decomposition Theorem [BBD82] and the fact that $\pi_{\mathcal{N}}$ is semi-small, it is a semi-simple perverse sheaf. Since it is also G -equivariant, its isotypic components are parametrized by some pairs (x, ρ) , up to G -conjugation, where x is a nilpotent element and ρ is an irreducible character of the finite group $A_G(x) =$

$C_G(x)/C_G(x)^0$. In both approaches using perverse sheaves, one can show that the endomorphism algebra of $\mathbb{K}\mathcal{K}_{\mathcal{N}}$ is isomorphic to the group algebra $\mathbb{K}W$. So the isotypic components are also parametrized by the irreducible characters of $\mathbb{K}W$, and this defines the correspondence.

In the Lusztig-Borho-MacPherson approach, one uses a functor of restriction to the nilpotent cone. Actually, one may give a direct construction of the simple perverse sheaf associated to an irreducible representation χ of W [Lus81]. One takes the intersection cohomology complex of the local system on the regular semisimple elements associated to χ , and restricts it to the nilpotent cone. Up to a shift by the rank of G , this is a simple G -equivariant perverse sheaf on \mathcal{N} , so it is associated with an intersection cohomology datum consisting of a pair (x, ρ) where x is a nilpotent element and ρ is an irreducible representation of $\mathbb{K}A_G(x)$ (corresponding to an irreducible local system on the orbit of x), up to G -conjugation, where $A_G(x)$ is the finite group $C_G(x)/C_G(x)^0$ of components of the centralizer of x in G . This gives a correspondence which differs from Springer's original parametrization by the sign character of W [Sho88, Proposition 17.7]. The proof that we get a simple perverse sheaf on \mathcal{N} by this restriction process really uses the Decomposition Theorem, and the result fails with characteristic ℓ coefficients.

On the other hand, the Fourier-Deligne transform still makes sense with characteristic ℓ coefficients, and sends a simple perverse sheaf to a simple perverse sheaf. We can still associate simple perverse sheaves to simple modules: if ϕ is a modular character of W , one takes the Fourier transform of the intersection cohomology complex of the local system on the regular semisimple elements associated to ϕ , which turns out to be supported by the nilpotent cone, as in characteristic zero. For the proof, we have to deal with non semi-simple perverse sheaves at some point, but it still works.

1.2. Decomposition matrices. For a moment, suppose W is any finite group, and choose a prime number ℓ . We also choose a sufficiently large finite extension \mathbb{K} of \mathbb{Q}_{ℓ} , with valuation ring \mathbb{O} and residue field \mathbb{F} . In representation theory, a triple such as $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ is called an ℓ -modular system. If E is a simple $\mathbb{K}W$ -module, we may choose a W -stable \mathbb{O} -lattice $E_{\mathbb{O}}$ in E , and form the $\mathbb{F}W$ -module $\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}}$. Though its isomorphism class depends on the choice of the lattice, it does have a well-defined class in the Grothendieck group $K_0(\mathbb{F}W)$. Thus, for each simple $\mathbb{F}W$ -module F , we have a well-defined multiplicity $d_{E,F}^W := [\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}} : F]$, which is called a decomposition number. The matrix

$$D^W := (d_{E,F}^W)_{E \in \text{Irr } \mathbb{K}W, F \in \text{Irr } \mathbb{F}W}$$

is called the (ℓ -modular) decomposition matrix of W . We refer the reader to [Ser67, Partie III] for more information.

When the ordinary characters are known, the determination of D^W is equivalent to the determination of the modular characters. Modular representations of finite groups are much less known than ordinary ones. For example, for the symmetric group \mathfrak{S}_n , we know the ordinary characters, but in characteristic ℓ , even if we know how to parametrize the simple modules [Jam76], we do not even know their dimensions explicitly in general.

When W is a Weyl group (as before), we will see that the determination of D^W can be translated into a geometrical or topological problem, thanks to the Springer

correspondence. Indeed, one can define a decomposition matrix $D^{\mathcal{N}}$ for the G -equivariant perverse sheaves on the nilpotent cone, just as in representation theory [Jut09], and we will see that D^W is a submatrix of $D^{\mathcal{N}}$. Thus to determine D^W , it would be enough to determine explicitly the modular Springer correspondence, and to compute the stalks of the modular intersection cohomology complexes. While the first task has now been achieved by [JLS, AHJR14a] and the present paper, the second one is probably extremely difficult, however one can hope to prove some qualitative results with these methods.

1.3. Some explicit results. The nilpotent orbits are naturally ordered by the inclusion of their closures, the smallest orbit being the trivial orbit $\mathcal{O}_{\text{triv}} = \{0\}$, and the largest orbit being the regular orbit \mathcal{O}_{reg} . Suppose G is simple. Then there is a unique minimal non-trivial orbit \mathcal{O}_{min} , and a unique maximal non-regular orbit, the subregular orbit $\mathcal{O}_{\text{subreg}}$. For G of adjoint type, we computed the decomposition numbers corresponding to the regular and subregular orbits, and to the minimal and trivial orbits in [Jut08, Jut09]. These results will be recalled.

A result of Kraft and Procesi [KP89] implies that the special pieces of the nilpotent cone are \mathbb{F}_ℓ -smooth as soon as $\ell \neq 2$, in classical types. We will deduce that some decomposition numbers, involving a special orbit and a smaller orbit in the same piece, are zero for $\ell \neq 2$.

Then we focus on type A . We determine explicitly the modular Springer correspondence in this case, and we show that James' row and column removal rule [Jam81] is a consequence of a smooth equivalence of nilpotent singularities obtained by Kraft and Procesi [KP81].

In the case of $G = SL_2$, we give the complete structure of the Springer and Grothendieck sheaves.

Finally, we determine explicitly the modular Springer correspondence for exceptional Weyl groups, using their decomposition matrices.

1.4. Related work. This paper contains the results of [Jut07, Chapters 6 and 8] (finally!), the results in Subsection 5.3 which were obtained at MSRI in 2008 during the program "Representation Theory of Finite Groups and Related Topics", and the explicit results for exceptional Weyl groups.

Several papers related to modular Springer theory have appeared in the mean time. In [Tre09], Rossmann's topological approach is followed to produce some version of the induction theorem for modular Springer representations.

In [Mau12], Mautner proves that, for $n \geq d$, the category of GL_d -equivariant perverse sheaves with \mathbb{E} coefficients on the nilpotent cone of \mathfrak{gl}_d is equivalent to the category of polynomial representations of GL_n over \mathbb{E} of degree d , using Lusztig's embedding of the nilpotent cone in the affine Grassmannian [Lus81], a map in the other direction at the level of stacks, and the geometric Satake correspondence [MV07]. Using the modular Springer correspondence, he then provides a geometric proof of Schur-Weyl duality, the Schur functor being described by homomorphisms from the Springer sheaf.

In [AM12], Achar and Mautner study the functor from the equivariant derived category of the nilpotent cone to itself given by Fourier transform on \mathfrak{g} followed by restriction to \mathcal{N} . They prove that it is an equivalence. In the case of GL_n it is a geometric version of Ringel duality, exchanging tilting sheaves (which are parity

sheaves [JMW14]) and projective sheaves. Moreover, they investigate examples in types B_2 and G_2 .

There is now a modular generalized Springer correspondence [AHJR13, AHJR14a, AHJR14c]. It is always defined (except possibly for type E_8 in characteristic 2); we can classify cuspidal pairs completely for classical types, and partially for exceptional types; we can determine the generalized correspondence explicitly for SL_n in any characteristic, for classical types in characteristic 2, and partially in exceptional types. In [JLS], we determine the non-generalized modular Springer correspondence for classical groups in odd characteristic. This last work uses basic sets for several orders (the Springer order, and some combinatorial orders which are known to be compatible with decomposition numbers), which justifies the introduction of the notion of “basic set datum”, where the order is part of the data.

The fact that the Fourier transform is an auto-equivalence makes it easy to show that the endomorphism algebra of the Springer sheaf is again the group algebra of the Weyl group. See 4.2.2 for a discussion of the approach by restriction to the nilpotent cone (either using the fact that both constructions differ by the sign character [AHJR14b], or directly with the Borel-Moore homology of the Steinberg variety [Ric13]).

We should also mention [AHR12] which, apart from being a continuation of [AH11], contains many compatibilities, notably of induction and restriction functors with respect to a modular Springer functor defined in terms of restriction to the nilpotent cone. By [AHJR14b], those compatibilities are valid also for the Fourier transform construction of the present paper.

Finally, the need to better understand the geometry of nilpotent cones to do further explicit calculation of stalks of modular IC sheaves was one of the motivations for [FJLS]. We describe generic singularities of nilpotent orbit closures for exceptional types (for classical types, this was done in [KP81, KP82]). Most of the time, we still find simple and minimal singularities, extending the applicability of the results in Section 6.

1.5. Outline. In Section 2, we recall some facts used in the sequel, mostly about modular perverse sheaves. In Section 3, we recall the definition and properties of the Fourier-Deligne transform that we need. In Section 4, we recall the background for Springer correspondence and we explain why we can still define such a correspondence in the modular case, using the Fourier-Deligne transform. In Section 5, we show that the decomposition matrix of a Weyl group is part of the decomposition matrix for equivariant perverse sheaves on the nilpotent cone, and that this defines a “Springer basic set” for the Weyl group. In Section 6, we give some decomposition numbers which can be obtained geometrically, using previous work. Section 7 is devoted to the study of modular Springer correspondence for the general linear group, including the row and column removal rule. In Section 8, we give explicit calculations of the structure of the Springer and Grothendieck sheaves for $G = SL_2$. Finally, in Section 9, we determine explicitly the modular Springer correspondence by displaying the decomposition matrices of the exceptional Weyl groups in the order given by Springer correspondence.

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2. PRELIMINARIES

Let k be a finite field \mathbb{F}_q or its algebraic closure. We will consider only k -varieties (separated k -schemes of finite type), and morphisms of k -schemes. We call p the characteristic of k .

Recall from 1.2 that $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ is an ℓ -modular system. We assume ℓ is different from p . Let X be a k -variety. For $\mathbb{E} = \mathbb{K}, \mathbb{O}, \mathbb{F}$, we have an abelian category of constructible \mathbb{E} -sheaves $\mathrm{Sh}(X, \mathbb{E})$. It contains the full subcategory of \mathbb{E} -local systems. For X connected, the latter is equivalent to the category of continuous representations of the fundamental group of X . We denote by $D_c^b(X, \mathbb{E})$ the bounded constructible derived category of \mathbb{E} -sheaves as defined by Deligne. In this setting we have Grothendieck's six operations: the tensor product $- \otimes_{\mathbb{E}}^L -$, the internal Hom functor $\mathrm{R}\underline{\mathrm{Hom}}(-, -)$, and, for $f : X \rightarrow Y$, the pairs of adjoint functors (f^*, f_*) and $(f_!, f^!)$ (all of these denote functors between derived categories). We denote by \mathbb{D}_X the Grothendieck-Verdier duality functor. We also have functors of extension of scalars $\mathbb{K}(-) := \mathbb{K} \otimes_{\mathbb{O}} (-)$ and of modular reduction $\mathbb{F}(-) := \mathbb{F} \otimes_{\mathbb{O}}^L (-)$.

The category ${}^p\mathcal{M}(X, \mathbb{E})$ of \mathbb{E} -perverse sheaves on X is the heart of the perverse t -structure on $D_c^b(X, \mathbb{E})$ for the middle perversity [BBD82]. In the case where \mathbb{E} is a field, this t -structure is stable by the duality, but for $\mathbb{E} = \mathbb{O}$ the duality exchanges this standard perverse t -structure with a dual perverse t -structure with heart ${}^{p+}\mathcal{M}(X, \mathbb{E})$. Over a point, the dual perverse sheaves are complexes of \mathbb{O} -modules with torsion-free H^0 and torsion H^1 (both finitely generated), all other cohomology groups being zero. We have perverse cohomology functors ${}^pH^m : D_c^b(X, \mathbb{E}) \rightarrow {}^p\mathcal{M}(X, \mathbb{E})$, and also ${}^{p+}H^m : D_c^b(X, \mathbb{E}) \rightarrow {}^{p+}\mathcal{M}(X, \mathbb{E})$ in case $\mathbb{E} = \mathbb{O}$. Distinguished triangles give rise to long exact sequences in perverse cohomology. We refer to [BBD82, §3.3] and [Jut09].

2.1. About intermediate extensions. Assume we are in a recollement situation [BBD82, §1.4]. So we have three t -categories (i.e. triangulated categories endowed with t -structures) \mathcal{D} , \mathcal{D}_U and \mathcal{D}_F related by six gluing functors, as follows:

$$(2.1) \quad \begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{D}_F & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{D}_U \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

satisfying certain axioms. The t -structure on \mathcal{D} is deduced from the t -structures on \mathcal{D}_U and \mathcal{D}_F by the recollement procedure. The hearts are denoted by \mathcal{C} , \mathcal{C}_U and \mathcal{C}_F . If T is any of the six gluing functors, we denote by pT the corresponding functor between the hearts, obtained as the composition of the inclusion of the heart of the source t -category, followed by T , and then by the perverse cohomology functor ${}^pH^0$ of the target t -category. Recall that i_* and j^* are t -exact and give rise to exact functors between the hearts (i_* is the inclusion of a thick subcategory, while j^* is the corresponding quotient functor).

For example ${}^{pi_*}{}^{pi^!}$ is the functor “largest subobject in ${}^{pi_*}\mathcal{C}_F$ ”, while ${}^{pi_*}{}^{pi^*}$ is the functor “largest quotient in ${}^{pi_*}\mathcal{C}_F$ ”. Note that we write pi_* to indicate that we

consider the functor between the hearts, although it is common to drop the p from the notation because the functor i_* is already t -exact. We will occasionally make this abuse of notation. The same remark applies to the functor ${}^p j^*$.

It follows from the axioms that there is a canonical morphism $j_! \rightarrow j_*$, and the latter factors through a canonical morphism ${}^p j_! \rightarrow {}^p j_*$. The intermediate extension functor ${}^p j_{!*}$ is defined as the image of this morphism. Over \mathbb{O} , there are also versions for p_+ of all those functors.

If \mathcal{S} (resp. $\mathcal{S}_U, \mathcal{S}_F$) denotes the set of (isomorphisms classes of) simple objects in \mathcal{C} (resp. $\mathcal{C}_U, \mathcal{C}_F$), then we have $\mathcal{S} = {}^p j_{!*} \mathcal{S}_U \cup {}^p i_* \mathcal{S}_F$.

The typical recollement situation arises when one considers an open immersion $j : U \rightarrow X$ with closed complement $i : F \rightarrow X$. The perverse t -structure on a stratified space is defined inductively by gluing shifts of natural t -structures on each stratum.

One bad feature of the intermediate extension functor ${}^p j_{!*}$ is that it is not exact. It may already happen with perverse sheaves in characteristic zero, however it is easy not to be aware of this problem because one usually applies this functor to a semisimple object. But with perverse sheaves with \mathbb{F} coefficients, the problem arises very often, and already in the Springer correspondence for GL_2 in characteristic 2. Nevertheless, the functor of intermediate extension enjoys some nice properties, that we will now recall. We refer to [Jut09] for more details, proofs and references.

Proposition 2.1. *The intermediate extension functor preserves monomorphisms and epimorphisms.*

Using the preceding result one can show the following.

Proposition 2.2. *Let A be an object of \mathcal{C}_U . Then we have*

$$\begin{aligned} \text{Soc } {}^p j_{!*} A &\simeq \text{Soc } {}^p j_* A \simeq {}^p j_{!*} \text{Soc } A, \\ \text{Top } {}^p j_{!*} A &\simeq \text{Top } {}^p j_! A \simeq {}^p j_{!*} \text{Top } A. \end{aligned}$$

Thus the intermediate extension functor preserves tops and socles. The second series of results is about composition multiplicities.

Proposition 2.3. *If B is a finite length object in \mathcal{C} , then ${}^p j^* B$ is of finite length in \mathcal{C}_U and we have*

$$[B : {}^p j_{!*} S] = [{}^p j^* B : S]$$

for all simple objects S in \mathcal{C}_U .

Since the three functors ${}^p j_!$, ${}^p j_{!*}$ and ${}^p j_*$ are extension functors (meaning that applying ${}^p j^*$ to them gives back the identity), we obtain the following corollary.

Corollary 2.4. *If A is an object in \mathcal{C}_U , then we have*

$$[{}^p j_! A : {}^p j_{!*} S] = [{}^p j_{!*} A : {}^p j_{!*} S] = [{}^p j_* A : {}^p j_{!*} S] = [A : S]$$

for all simple objects S in \mathcal{C}_U , whenever those multiplicities are defined (i.e. when the objects involved have finite length).

So the intermediate extension functor preserves composition multiplicities. This will imply that, in the context of perverse sheaves, the intermediate extension functor preserves decomposition numbers. Let us first recall from [Jut09] that, in an open/closed situation, starting with a torsion-free object A in ${}^p \mathcal{M}(U, \mathbb{O})$, we can define a priori nine extensions of $\mathbb{F}A$ with the functors $\mathbb{F} {}^p j_?$, ${}^p j_? \mathbb{F}$ and $\mathbb{F} {}^{p+} j_?$ where

$? \in \{!, !*, *\}$. Out of those, ${}^p j_!$ and ${}^{p+} j_*$ might fail to be perverse but the other seven are automatically perverse. The three involving $!*$ ones are particularly interesting as they are the modular reduction of the \mathcal{IC} , the \mathcal{IC} of the modular reduction, and the modular reduction of the dual \mathcal{IC} . Since they are all extensions of $\mathbb{F}A$, using the proposition we get the following.

Corollary 2.5. *If A is a torsion-free object in ${}^p\mathcal{M}(X, \mathbb{O})$, then*

$$[\mathbb{F} {}^p j_{!*} A : {}^p j_{!*} S] = [{}^p j_{!*} \mathbb{F} A : {}^p j_{!*} S] = [\mathbb{F} {}^{p+} j_{!*} A : {}^p j_{!*} S] = [\mathbb{F} A : S].$$

Now let X be a stratified algebraic variety. Recall from [Jut09] that we can define decomposition numbers for perverse sheaves by

$$d_{(\mathcal{O}, \mathcal{L}), (\mathcal{O}', \mathcal{L}')}^X := [\mathbb{F} \mathcal{IC}(\overline{\mathcal{O}}, \mathcal{L}_{\mathbb{O}}) : \mathcal{IC}(\overline{\mathcal{O}'}, \mathcal{L}')]]$$

where \mathcal{O} and \mathcal{O}' are two strata, \mathcal{L} is a \mathbb{K} -local system on \mathcal{O} with some integral form $\mathcal{L}_{\mathbb{O}}$, and \mathcal{L}' is an \mathbb{F} -local system on \mathcal{O}' . For simplicity, we assume that the representations of the fundamental groups corresponding to the considered local systems factor through a finite group, so that it is clear that a stable lattice exists and that decomposition numbers are well defined — this will be the case for the nilpotent cone stratified by the group orbits. (However, we note that a finite dimensional continuous representation over \mathbb{K} of a profinite group like the étale fundamental group always has a stable lattice.) The preceding corollary implies that decomposition numbers between \mathcal{IC} 's with the same support are just decomposition numbers for (a finite quotient of) the fundamental group of the stratum, as stated below.

Corollary 2.6. *Let \mathcal{O} be a stratum in X and let \mathcal{L} and \mathcal{L}' be two local systems on \mathcal{O} corresponding to irreducible representations ρ and ρ' of $\pi_1(\mathcal{O})$ which factor through a finite quotient H . Then*

$$d_{(\mathcal{O}, \mathcal{L}), (\mathcal{O}, \mathcal{L}')}^X = d_{\rho, \rho'}^H.$$

We have a natural partial order on the strata of X defined by $\mathcal{O} \geq \mathcal{O}'$ if $\overline{\mathcal{O}} \supset \mathcal{O}'$. For decomposition numbers involving different strata, one can make the following easy observation.

Proposition 2.7. *If some decomposition number $d_{(\mathcal{O}, \mathcal{L}), (\mathcal{O}', \mathcal{L}')}^X$ is nonzero, then $\mathcal{O}' \leq \mathcal{O}$.*

Proof. If $\mathcal{L}_{\mathbb{O}}$ is an integral form of the local system \mathcal{L} , then $\mathcal{IC}(\overline{\mathcal{O}}, \mathcal{L}_{\mathbb{O}})$ is an integral form of $\mathcal{IC}(\overline{\mathcal{O}}, \mathcal{L})$, and is clearly supported by $\overline{\mathcal{O}}$, hence $\mathbb{F} \mathcal{IC}(\overline{\mathcal{O}}, \mathcal{L})$ is also supported by $\overline{\mathcal{O}}$. Now ${}^p i_* {}^p \mathcal{M}(\overline{\mathcal{O}}, \mathbb{F})$, where $i : \overline{\mathcal{O}} \rightarrow X$ is the closed inclusion, is a Serre subcategory of ${}^p\mathcal{M}(X, \mathbb{F})$, hence the result. \square

Let us assume that X is a G -variety with finitely many orbits, and let us consider only G -equivariant local systems and perverse sheaves. The G -equivariant local systems on some orbit \mathcal{O} correspond to finite dimensional representations of the finite group $A_G(\mathcal{O}) = C_G(x)/C_G^0(x)$ where x is some point in \mathcal{O} . Corollary 2.6 and Proposition 2.7 mean that the decomposition matrix for G -equivariant perverse sheaves on X is block triangular, and the diagonal blocks are decomposition matrices for the groups $A_G(\mathcal{O})$.

We have used the fact that an \mathcal{IC} complex over \mathbb{K} has some integral form, namely an \mathcal{IC} complex over \mathbb{O} , with the same support. In the next subsection we will see a better statement: for any perverse sheaf on some closed subvariety, any integral form is supported by that closed subset.

2.2. Integral forms and supports. In this paragraph we prove a result which we will need in Section 4, and which would have found its place in [Jut09, §2.5]. We consider a k -variety X with an open subvariety $j : U \rightarrow X$ and closed complement $i : Z \rightarrow X$. So we are in a recollement situation.

Proposition 2.8. *Suppose A is a \mathbb{K} -perverse sheaf on X supported on Z , and let $A_{\mathbb{O}}$ be an integral form of A , that is, a torsion-free \mathbb{O} -perverse sheaf on X such that $\mathbb{K}A_{\mathbb{O}} \simeq A$. Then $A_{\mathbb{O}}$ is also supported by Z .*

Proof. First, we have $\mathbb{K}j^*A_{\mathbb{O}} = j^*A = 0$, hence $j^*A_{\mathbb{O}}$ is torsion. It follows that ${}^pj_!j^*A_{\mathbb{O}}$ is torsion as well. Now, from the distinguished triangle

$$j_!j^*A_{\mathbb{O}} \rightarrow A_{\mathbb{O}} \rightarrow i_*i^*A_{\mathbb{O}} \rightsquigarrow$$

we get the following perverse cohomology exact sequence:

$${}^pj_!j^*A_{\mathbb{O}} \rightarrow A_{\mathbb{O}} \rightarrow {}^pi_*{}^pi^*A_{\mathbb{O}} \rightarrow 0$$

since $j_!$ is right t -exact. Now, the first map must be zero as it goes from a torsion object to a torsion-free object. Thus $A_{\mathbb{O}} \simeq {}^pi_*{}^pi^*A_{\mathbb{O}}$ is supported by Z . \square

2.3. Small and semi-small morphisms. Let us recall the notions of semi-small and small morphisms, and a proposition which shows their usefulness (the usual proof applies).

Definition 2.9. A morphism $\pi : \tilde{X} \rightarrow X$ is *semi-small* if there is a stratification \mathfrak{X} of X such that the for all strata S in \mathfrak{X} , and for all closed points s in S , we have $\dim \pi^{-1}(s) \leq \frac{1}{2} \operatorname{codim}_X(S)$. If moreover these inequalities are strict for all strata of positive codimension, we say that π is *small*.

Proposition 2.10. *Let $\pi : \tilde{X} \rightarrow X$ be a surjective, proper and separable morphism, with \tilde{X} smooth irreducible of dimension d . Let \mathcal{L} be an \mathbb{E} -local system on \tilde{X} . Let us consider the complex $K = \pi_*\mathcal{L}[d]$.*

- (i) *If π is semi-small, then π is generically finite and K is p -perverse.*
- (ii) *If π is small, then $K = {}^pj_!{}^pj^*K$ for any inclusion $j : U \rightarrow X$ of a smooth open dense subvariety over which π is étale.*

2.4. Rational smoothness, ℓ -smoothness. Suppose X is an irreducible variety. If $j : V \rightarrow X$ the inclusion of a smooth open dense subvariety and \mathcal{L} is a local system on V , then we denote by $\mathcal{IC}(X, \mathcal{L})$ the intermediate extension ${}^pj_{!*}(\mathcal{L}[\dim X])$. We say that X is \mathbb{E} -smooth if $\mathcal{IC}(X, \mathbb{E})$ is reduced to $\underline{\mathbb{E}}_X[\dim X]$. When $\mathbb{E} = \mathbb{O}$, we require this condition for both perversities, p and p_+ . This property ensures that X satisfies Poincaré duality with \mathbb{E} coefficients.

Proposition 2.11. *Let H be a finite group of order prime to ℓ . If X is an \mathbb{F}_{ℓ} -smooth H -variety, then X/H is also \mathbb{F}_{ℓ} -smooth.*

3. FOURIER-DELIGNE TRANSFORM

We will recall well known facts about the Fourier-Deligne transform (see [Lau87]), pointing out that it makes sense with coefficients in \mathbb{K} , \mathbb{O} or \mathbb{F} . It is a pleasant exercise to fill in the details in Laumon's elegant exposition. One may refer to [Jut07, Chapter 5] for details.

In this section, $k = \mathbb{F}_q$. Let us assume that \mathbb{E}^{\times} contains a primitive root of unity of order p . We fix a non-trivial character $\psi : \mathbb{F}_p \rightarrow \mathbb{E}^{\times}$, that is, a primitive root

$\psi(1)$ of order p in \mathbb{E}^\times . Composing with $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$, we get a character of \mathbb{F}_q . Let \mathcal{L}_ψ be the locally constant \mathbb{E} -sheaf of rank 1 on the additive group \mathbb{G}_a associated to ψ (the corresponding Artin-Schreier local system).

Let S be a variety, and $E \xrightarrow{\pi} S$ a vector bundle of constant rank $r \geq 1$. We denote by $E' \xrightarrow{\pi'} S$ its dual vector bundle, by $\mu : E \times_S E' \rightarrow \mathbb{A}^1$ the canonical pairing, and by $\mathrm{pr} : E \times_S E' \rightarrow E$ and $\mathrm{pr}' : E \times_S E' \rightarrow E'$ the canonical projections. So we have the following diagram.

$$\begin{array}{ccccc}
 & E \times_S E' & \xrightarrow{\mu} & \mathbb{A}^1 & \\
 \mathrm{pr} \swarrow & & & & \searrow \mathrm{pr}' \\
 E & & \square & & E' \\
 \pi \searrow & & & & \swarrow \pi' \\
 & S & & &
 \end{array}$$

Definition 3.1. The Fourier-Deligne transform for $E \xrightarrow{\pi} S$, associated to the character ψ , is the triangulated functor

$$\mathcal{F}_\psi : D_c^b(E, \mathbb{E}) \longrightarrow D_c^b(E', \mathbb{E})$$

defined by

$$\mathcal{F}_\psi(K) = \mathrm{pr}'_!(\mathrm{pr}^* K \otimes_{\mathbb{E}} \mu^* \mathcal{L}_\psi)[r]$$

In the sequel, we will drop the indices ψ from the notations \mathcal{F}_ψ and \mathcal{L}_ψ when no confusion may arise.

Let $E'' \xrightarrow{\pi''} S$ be the bidual vector bundle of $E \xrightarrow{\pi} S$ and $a : E \xrightarrow{\sim} E''$ the S -isomorphism defined by $a(e) = -\mu(e, -)$ (that is, the opposite of the canonical S -isomorphism). We will denote by $\sigma : S \rightarrow E$, $\sigma' : S \rightarrow E'$ and $\sigma'' : S \rightarrow E''$ the respective null sections of π , π' and π'' . Finally, we will denote by $s : E \times_S E \rightarrow E$ the addition of the vector bundle $E \xrightarrow{\pi} S$ and by $-1_E : E \rightarrow E$ the opposite for this addition.

The following Proposition is the analogue of the fact that the Fourier transform of the constant function is a Dirac distribution supported at the origin in classical Fourier analysis. By the function/sheaf dictionary, this becomes a functorial isomorphism, to which we will refer as (DIRAC). It will be used to prove that the pair corresponding to the trivial character 1_W is the trivial pair $(0, 1)$ consisting of the zero nilpotent orbit with the trivial local system, both in the ordinary and modular cases.

Proposition 3.2. *We have a functorial isomorphism*

$$(\text{DIRAC}) \quad \mathcal{F}(\pi^* L[r]) \simeq \sigma'_* L(-r)$$

for all objects L in $D_c^b(S, \mathbb{E})$.

The following fundamental result says that there is an inverse Fourier transform. We will use it to see that the Springer correspondence is injective.

Theorem 3.3. *Let \mathcal{F}' be the Fourier-Deligne transform, associated to the character ψ , of the vector bundle $E' \xrightarrow{\pi'} S$. Then we have a functorial isomorphism*

$$(\text{INV}) \quad \mathcal{F}' \circ \mathcal{F}(K) \simeq a_* K(-r)$$

for all objects K in $D_c^b(E, \mathbb{E})$.

Corollary 3.4. *The triangulated functor \mathcal{F} is an equivalence of triangulated categories from $D_c^b(E, \mathbb{E})$ to $D_c^b(E', \mathbb{E})$, with quasi-inverse $a^* \mathcal{F}'(-)(r)$.*

We will also need the following result, which describes how the Fourier-Deligne transform behaves with respect to morphisms of vector bundles.

Theorem 3.5. *Let $f : E_1 \rightarrow E_2$ be a morphism of vector bundles over S , with constant ranks r_1 and r_2 respectively, and let $f' : E'_2 \rightarrow E'_1$ denote the transposed morphism. Then we have a functorial isomorphism*

$$(MOR) \quad \mathcal{F}_2(f_! K_1) \simeq f'^* \mathcal{F}_1(K_1)[r_2 - r_1]$$

for K_1 in $D_c^b(E_1, \mathbb{E})$, where \mathcal{F}_1 and \mathcal{F}_2 denote the Fourier-Deligne transforms for E_1 and E_2 .

With this result, one can generalize Proposition 3.2 (DIRAC) to any sub-vector bundle:

Proposition 3.6. *Let $i : F \hookrightarrow E$ be a sub-vector bundle over S , with constant rank r_F . We denote by $i^\perp : F^\perp \hookrightarrow E'$ the orthogonal of F in E' . Then we have a canonical isomorphism*

$$(SUB) \quad \mathcal{F}(i_* \mathbb{E}_F[r_F]) \simeq i_*^\perp \mathbb{E}_{F^\perp}(-r_F)[r - r_F]$$

Now we consider how the Fourier-Deligne transform behaves with respect to a base change. We will need (BC_!) to prove that the Fourier-Deligne transform of the Grothendieck sheaf is the Springer sheaf.

Proposition 3.7. *Let $f : S_1 \rightarrow S$ be an \mathbb{F}_q -morphism of finite type. We have a base change diagram:*

$$\begin{array}{ccccc}
 E_1 \times_{S_1} E'_1 & & \xrightarrow{\mu_1} & & \mathbb{A}^1 \\
 \swarrow \text{pr}_1 & \searrow \text{pr}'_1 & \searrow F & \Delta & \searrow \mu \\
 E_1 & & E'_1 & \xrightarrow{f_{E'}} & E \times_S E' \\
 \searrow \pi_1 & \swarrow \pi'_1 & \searrow f_E & \swarrow \text{pr} & \searrow \text{pr}' \\
 & S_1 & & E & E' \\
 & \searrow f & \searrow \pi & \searrow \pi' & \\
 & & S & &
 \end{array}$$

Let \mathcal{F}_1 denotes the Fourier-Deligne transform associated to $E_1 \xrightarrow{\pi_1} S$. Then we have functorial isomorphisms

$$(BC^*) \quad \mathcal{F}_1(f_E^* K) \simeq f_{E'}^* \mathcal{F}(K)$$

$$(BC_!) \quad \mathcal{F}(f_{E!} K_1) \simeq f_{E'!} \mathcal{F}_1(K_1)$$

for K in $D_c^b(E, \mathbb{E})$ and K_1 in $D_c^b(E_1, \mathbb{E})$.

The next result shows that the Fourier-Deligne transform preserves equivariance.

Proposition 3.8. *Let G be a smooth affine group scheme over S , acting linearly on the vector bundle $E \xrightarrow{\pi} S$, let K and L be two objects in $D_c^b(E, \mathbb{E})$, and let M be an object in $D_c^b(G, \mathbb{E})$. We denote by $m : G \times_S E \rightarrow E$ the action of G on E ,*

and by $m' : G \times_S E' \rightarrow E'$ the contragredient action, defined by $m'(g, e') = {}^t g^{-1}.e'$. Then each isomorphism

$$m^* K \simeq M \boxtimes_S^{\mathbb{L}} L$$

in $D_c^b(G \times_S E, \mathbb{E})$ induces canonically an isomorphism

$$(G\text{-EQ}) \quad m'^* \mathcal{F}(K) \simeq M \boxtimes_S^{\mathbb{L}} \mathcal{F}(L)$$

in $D_c^b(G \times_S E', \mathbb{E})$.

One could have used $\mathrm{pr}'_!$ instead of pr'_* to define an a priori different notion of Fourier-Deligne transform. The following fundamental result shows that both are canonically isomorphic [KL85, Appendice 2.4].

Theorem 3.9. *For any object K in $D_c^b(E, \mathbb{E})$, the support forgetting morphism*

$$(SUPP) \quad \mathrm{pr}'_! (\mathrm{pr}^* K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \longrightarrow \mathrm{pr}'_* (\mathrm{pr}^* K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L})$$

is an isomorphism.

This theorem implies the powerful property that Fourier transform commutes with duality.

Theorem 3.10. *We have a functorial isomorphism*

$$\mathrm{RHom}(\mathcal{F}_\psi(K), \pi^! L) \simeq \mathcal{F}_{\psi^{-1}}(\mathrm{RHom}(K, \pi^! L))(r)$$

for (K, L) in $D_c^b(E, \mathbb{E})^{\mathrm{op}} \times D_c^b(S, \mathbb{E})$.

Remember that, if X is a variety, we denote by $\mathcal{D}_{X, \mathbb{E}}$ the duality functor of $D_c^b(X, \mathbb{E})$. If $a : X \rightarrow \mathrm{Spec} k$ is the structural morphism, we denote by $D_{X, \mathbb{E}}$ the dualizing complex $a^! \mathbb{E}$.

Corollary 3.11. *We have a functorial isomorphism*

$$\mathcal{D}_{E', \mathbb{E}}(\mathcal{F}_\psi(K)) \simeq \mathcal{F}_{\psi^{-1}}(\mathcal{D}_{E, \mathbb{E}}(K))(r)$$

for K in $D_c^b(E, \mathbb{E})^{\mathrm{op}}$.

Theorem 3.12. *\mathcal{F} maps ${}^p \mathcal{M}(E, \mathbb{E})$ onto ${}^p \mathcal{M}(E', \mathbb{E})$. The functor*

$$(EQUIV) \quad \mathcal{F} : {}^p \mathcal{M}(E, \mathbb{E}) \longrightarrow {}^p \mathcal{M}(E', \mathbb{E})$$

is an equivalence of abelian categories, with quasi-inverse $a^ \mathcal{F}'(-)(r)$.*

Moreover, by Proposition 3.8 (G-EQ), \mathcal{F} sends G -equivariant perverse sheaves on G -equivariant perverse sheaves (take the constant perverse sheaf on G for M).

Corollary 3.13. *Suppose $\mathbb{E} = \mathbb{K}$ or \mathbb{F} . Then \mathcal{F} transforms simple \mathbb{E} -perverse sheaves on E into simple \mathbb{E} -perverse sheaves on E' .*

This will play a crucial role in the construction of Springer correspondence. Again, we have a version with G -equivariant perverse sheaves.

4. SPRINGER CORRESPONDENCE

4.1. The geometric context.

4.1.1. *Notation.* Let G be a split connected semisimple linear algebraic group of rank r over k . Let us fix a Borel subgroup B of G , with unipotent radical U , and a maximal torus T contained in B . We denote by \mathfrak{g} , \mathfrak{b} , \mathfrak{u} and \mathfrak{t} the corresponding Lie algebras. The characters of T form a free abelian group $X(T)$ of rank r . The Weyl group $W = N_G(T)/T$ acts as a reflection group on $V = \mathbb{Q} \otimes_{\mathbb{Z}} X(T)$.

Let $\Phi \subset X(T)$ be the root system of (G, T) , Φ^+ the set of positive roots defined by B , and Δ the corresponding basis. We denote by ν_G (or just ν) the cardinality of Φ^+ . Then $\dim G = 2\nu + r$, $\dim B = \nu + r$, $\dim T = r$ and $\dim U = \nu$.

4.1.2. *The finite quotient map.* Let $\phi : \mathfrak{t} \rightarrow \mathfrak{t}/W$ be the quotient map, corresponding to the inclusion $k[\mathfrak{t}]^W \hookrightarrow k[\mathfrak{t}]$. It is finite and surjective. For $t \in \mathfrak{t}$, we will also denote $\phi(t)$ by \bar{t} .

Let us assume that p is not a torsion prime for \mathfrak{g} . Then $k[\mathfrak{t}]^W = k[\phi_1, \dots, \phi_r]$ for some algebraically independent homogeneous polynomials ϕ_1, \dots, ϕ_r whose degrees $d_1 \leq \dots \leq d_r$ are well defined and called *characteristic degrees* [Dem73, Théorème 3]. We have $d_i = m_i + 1$, where the m_i are the exponents of W (determining the eigenvalues of a Coxeter element) [Bou68, Chap. VI, §6.2]. The quotient space \mathfrak{t}/W can be identified with \mathbb{A}^r and ϕ with (ϕ_1, \dots, ϕ_r) .

For example, if $G = SL_n$, we can identify \mathfrak{t} with the hyperplane $\{(x_1, \dots, x_n) \mid x_1 + \dots + x_n = 0\}$ of k^n , and for ϕ_i ($1 \leq i \leq r = n - 1$) we can take the $(i + 1)$ st elementary symmetric polynomial σ_{i+1} on k^n , restricted to this hyperplane (σ_1 does not appear, since its restriction vanishes).

4.1.3. *The adjoint quotient.* Assume that $p > 2$ if G has a component of type C_m , $m \geq 1$. Then by [CR10] the Chevalley restriction theorem holds: the restriction map $k[\mathfrak{g}]^G \rightarrow k[\mathfrak{t}]^W$ is an isomorphism. Recall that we assume moreover that p is not a torsion prime for G , so that the preceding paragraph applies. Let χ_i , $1 \leq i \leq r$, be the element of $k[\mathfrak{g}]^G$ restricting to ϕ_i . Then the χ_i are homogeneous algebraically independent polynomials of degrees $d_1 \leq \dots \leq d_r$. Hence we have a morphism $\chi = (\chi_1, \dots, \chi_r) : \mathfrak{g} \rightarrow \mathfrak{g}/G \simeq \mathfrak{t}/W \simeq \mathbb{A}^r$. It is called the Steinberg map, or the adjoint quotient.

The morphism χ has been extensively studied (see [Slo80b] and the references therein). First, it is flat, and its schematic fibers are irreducible, reduced and normal complete intersections, of codimension r in \mathfrak{g} . If $t \in \mathfrak{t}$, let $\mathfrak{g}_{\bar{t}}$ be the fiber $\chi^{-1}(\bar{t})$. It is the union of finitely many adjoint orbits. It contains exactly one orbit of regular elements, which is open and dense in $\mathfrak{g}_{\bar{t}}$, and whose complement has codimension ≥ 2 in $\mathfrak{g}_{\bar{t}}$. This regular orbit is exactly the smooth locus of $\mathfrak{g}_{\bar{t}}$. So \mathfrak{t}/W parametrizes the orbits of regular elements. The fiber $\mathfrak{g}_{\bar{t}}$ also contains exactly one orbit of semisimple elements, the orbit of t , which is the only closed orbit in $\mathfrak{g}_{\bar{t}}$, and which lies in the closure of every other orbit in $\mathfrak{g}_{\bar{t}}$. In fact, χ can be interpreted as the map which sends x to the intersection of the orbit of x_s with \mathfrak{t} , which is a W -orbit.

For example, for $G = SL_n$, we can define the $\chi_i : \mathfrak{sl}_n \rightarrow k$ by the formula

$$\det(\xi - x) = \xi^n + \sum_{i=1}^{n-1} (-1)^{i+1} \chi_i(x) \xi^{n-1-i} \in k[\xi]$$

for $x \in \mathfrak{sl}_n$. So $\chi(x)$ can be interpreted as the characteristic polynomial of x . Restricting χ_i to \mathfrak{t} , we recover $\phi_i = \sigma_{i+1}$.

4.1.4. *Springer's resolution of the nilpotent cone.* Let \mathcal{N} be the closed subvariety of \mathfrak{g} consisting of its nilpotent elements. It is the fiber $\mathfrak{g}_0 = \chi^{-1}(0)$. In particular, it is a complete intersection in \mathfrak{g} , given by the equations $\chi_1(x) = \cdots = \chi_r(x) = 0$. It is singular. We are going to describe Springer's resolution of the nilpotent cone.

The set \mathcal{B} of Borel subalgebras of \mathfrak{g} is a homogeneous space under G , in bijection with G/B , since the normalizer of \mathfrak{b} in G is B . Hence \mathcal{B} is endowed with a structure of smooth projective variety, of dimension ν .

Let $\tilde{\mathcal{N}} = \{(x, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}'\}$. It is a smooth variety of dimension $2\nu = \dim \mathcal{N}$: the second projection makes it a vector bundle over \mathcal{B} , which can be identified with $G \times^B \mathfrak{u}$ via $g *^B x \mapsto (\text{Ad}(g)(x), \text{Ad}(g)(\mathfrak{b}))$. It can be further identified with the cotangent bundle $T^*\mathcal{B}$, since $T\mathcal{B} = T(G/B) = G \times^B (\mathfrak{g}/\mathfrak{b})$ and $\mathfrak{u} = \mathfrak{b}^\perp$. Now let $\pi_{\mathcal{N}} : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ be the first projection. Since $\tilde{\mathcal{N}}$ is closed in $\mathcal{N} \times \mathcal{B}$ and \mathcal{B} is projective, the morphism $\pi_{\mathcal{N}}$ is projective. Moreover, it is an isomorphism over the open dense subvariety of \mathcal{N} consisting of the regular nilpotent elements, so $\pi_{\mathcal{N}}$ is indeed a resolution of \mathcal{N} .

4.1.5. *Grothendieck's simultaneous resolution of the adjoint quotient.* In the last paragraph, we have seen the resolution of the fiber $\chi^{-1}(0)$. We are now going to explain Grothendieck's simultaneous resolution, which gives resolutions for all the fibers of χ simultaneously.

Let $\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}') \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}'\}$ and let $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ be the first projection. With the second projection, $\tilde{\mathfrak{g}}$ is a vector bundle over \mathcal{B} , isomorphic to $G \times^B \mathfrak{b}$ via $g *^B x \mapsto (\text{Ad}(g)(x), \text{Ad}(g)(\mathfrak{b}))$. Hence $\tilde{\mathfrak{g}}$ is a smooth variety of dimension $2\nu + r = \dim \mathfrak{g}$. Finally, we define θ as the composition $\tilde{\mathfrak{g}} \simeq G \times^B \mathfrak{b} \rightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \xrightarrow{\sim} \mathfrak{t}$. Then the commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\pi} & \mathfrak{g} \\ \theta \downarrow & & \downarrow \chi \\ \mathfrak{t} & \xrightarrow{\phi} & \mathfrak{t}/W \end{array}$$

is a simultaneous resolution of the singularities of the flat morphism χ . That is, θ is smooth, ϕ is finite surjective, π is proper, and π induces a resolution of singularities $\theta^{-1}(t) \rightarrow \chi^{-1}(\phi(t))$ for all $t \in \mathfrak{t}$.

4.2. **Springer correspondence for $\mathbb{E}W$.** If $\mathbb{E} = \mathbb{K}$ or \mathbb{F} , let $\mathfrak{P}_{\mathbb{E}}$ denote the set of pairs $(\mathcal{O}, \mathcal{L})$, where \mathcal{O} is some nilpotent orbit and \mathcal{L} is an irreducible G -equivariant local system on \mathcal{O} . Those pairs index the simple G -equivariant perverse sheaves on \mathcal{N} . So their classes form a basis of the Grothendieck group of the category of G -equivariant perverse sheaves on \mathcal{N} . In formulas, we have

$$K_0({}^p\mathcal{M}(\mathcal{N}, \mathbb{E})) \simeq \bigoplus_{\mathcal{O}} K_0(\mathbb{E}A_G(\mathcal{O})) \simeq \bigoplus_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}_{\mathbb{E}}} \mathbb{Z}[\mathcal{IC}(\overline{\mathcal{O}}, \mathcal{L})].$$

The aim of this subsection is to define a Springer correspondence for $\mathbb{E}W$, that is an injection $\Psi_{\mathbb{E}} : \text{Irr } \mathbb{E}W \rightarrow \mathfrak{P}_{\mathbb{E}}$, valid in both cases $\mathbb{E} = \mathbb{K}$ and $\mathbb{E} = \mathbb{F}$.

4.2.1. *The perverse sheaves \mathcal{K}_{rs} , \mathcal{K} and $\mathcal{K}_{\mathcal{N}}$.* Let us consider the following commutative diagram with cartesian squares:

$$\begin{array}{ccccc}
\tilde{\mathfrak{g}}_{\text{rs}} & \xrightarrow{\tilde{j}_{\text{rs}}} & \tilde{\mathfrak{g}} & \xleftarrow{i_{\tilde{\mathcal{N}}}} & \tilde{\mathcal{N}} \\
\pi_{\text{rs}} \downarrow & \square_{\text{rs}} & \downarrow \pi & \square_{\mathcal{N}} & \downarrow \pi_{\mathcal{N}} \\
\mathfrak{g}_{\text{rs}} & \xrightarrow{j_{\text{rs}}} & \mathfrak{g} & \xleftarrow{i_{\mathcal{N}}} & \mathcal{N}
\end{array}$$

Let us define the complex

$$\mathcal{K} := \pi_* \mathbb{O}_{\tilde{\mathfrak{g}}} [2\nu + r].$$

Note that $\pi_* = \pi_!$ since π is proper. Let \mathbb{E} be \mathbb{K} , \mathbb{O} or \mathbb{F} . Since modular reduction commutes with direct images, we have $\mathbb{E}\mathcal{K} = \pi_* \mathbb{E}_{\tilde{\mathfrak{g}}} [2\nu + r]$. By the proper base change theorem, the fiber at a point x in \mathfrak{g} of $\mathbb{E}\mathcal{K}$ is given by $(\mathbb{E}\mathcal{K})_x = \text{R}\Gamma(\mathcal{B}_x, \mathbb{E})$.

Let $\mathcal{K}_{\text{rs}} = j_{\text{rs}}^* \mathcal{K}$ and $\mathcal{K}_{\mathcal{N}} = i_{\mathcal{N}}^* \mathcal{K}[-r]$. By the proper base change theorem and the commutation between modular reduction and inverse images, we have

$$\begin{aligned}
\mathbb{E}\mathcal{K}_{\text{rs}} &= j_{\text{rs}}^* \mathbb{E}\mathcal{K} = \pi_{\text{rs}*} \mathbb{E}_{\tilde{\mathfrak{g}}_{\text{rs}}} [2\nu + r], \\
\mathbb{E}\mathcal{K}_{\mathcal{N}} &= i_{\mathcal{N}}^* \mathbb{E}\mathcal{K}[-r] = \pi_{\mathcal{N}*} \mathbb{E}_{\tilde{\mathcal{N}}} [2\nu].
\end{aligned}$$

The morphism π is separable, proper and small, hence $\mathbb{E}\mathcal{K}$ is an intersection cohomology complex by Proposition 2.10. The morphism π_{rs} obtained after the base change j_{rs} is a Galois finite étale covering, with Galois group W . In particular, W is a quotient of the étale fundamental group of \mathfrak{g}_{rs} . If E is a representation of W (over \mathbb{K} , \mathbb{O} or \mathbb{F}), we will simply denote by \underline{E} the associated local system on \mathfrak{g}_{rs} , shifted by the dimension $2\nu + r$. Then we have $\mathbb{E}\mathcal{K} = j_{\text{rs}!} \mathbb{E}\mathcal{K}_{\text{rs}} = j_{\text{rs}!} \mathbb{E}\underline{W}$. Note that, if $\mathbb{E} = \mathbb{O}$, we have ${}^{p+}j_{\text{rs}!} \mathcal{K}_{\text{rs}} = \mathcal{D}_{\mathfrak{g}, \mathbb{O}}({}^p j_{\text{rs}!} \mathcal{K}_{\text{rs}}) = \mathcal{D}_{\mathfrak{g}, \mathbb{O}}(\mathcal{K}) = \mathcal{K}$ so it does not matter whether we use p or p_+ (we have used the fact that the regular representation is self-dual, and that \mathcal{K} is self-dual because π is proper and $\tilde{\mathfrak{g}}$ is smooth).

Thus the endomorphism algebra of $\mathbb{E}\mathcal{K}_{\text{rs}}$ is the group algebra $\mathbb{E}W$. Since the functor $j_{\text{rs}!}$ is fully faithful, it induces an isomorphism $\text{End}(\mathcal{K}_{\text{rs}}) = \mathbb{E}W \xrightarrow{\sim} \text{End}(\mathcal{K})$. In particular, we have an action of $\mathbb{E}W$ on the stalks $\mathcal{H}_x^i(\mathbb{E}\mathcal{K}) = H^{i+2\nu+r}(\mathcal{B}_x, \mathbb{E})$.

When $\mathbb{E} = \mathbb{K}$, the group algebra $\mathbb{K}W$ is semisimple, and so are the perverse sheaves

$$\mathbb{K}\mathcal{K}_{\text{rs}} \simeq \bigoplus_{E \in \text{Irr } \mathbb{K}W} \underline{E}^{\dim E}$$

and

$$\mathbb{K}\mathcal{K} \simeq \bigoplus_{E \in \text{Irr } \mathbb{K}W} (j_{\text{rs}!} \underline{E})^{\dim E}.$$

If ℓ does not divide the order of the Weyl group W , then we have a similar decomposition for $\mathbb{E} = \mathbb{O}$ or \mathbb{F} . However, we are mostly interested in the case where ℓ divides $|W|$. Then $\mathbb{F}\mathcal{K}_{\text{rs}}$ and $\mathbb{F}\mathcal{K}$ are not semisimple. More precisely, we have decompositions

$$\begin{aligned}
\mathbb{O}W &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} P_F^{\dim F} \\
\mathbb{F}W &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} (\mathbb{F}P_F)^{\dim F}
\end{aligned}$$

where P_F is a projective indecomposable $\mathbb{O}W$ -module such that $\mathbb{F}P_F$ is a projective cover of F . Besides, $\mathbb{F}P_F$ has top and socle isomorphic to F .

Hence we have a similar decomposition for \mathcal{K}_{rs} , and its modular reduction:

$$\begin{aligned}\mathcal{K}_{\text{rs}} &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} \underline{P}_F^{\dim F}, \\ \mathbb{F}\mathcal{K}_{\text{rs}} &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} (\underline{\mathbb{F}P}_F)^{\dim F}.\end{aligned}$$

These are decompositions into indecomposable summands, and the indecomposable summand $\underline{\mathbb{F}P}_F$ has top and socle isomorphic to \underline{F} . By Proposition 2.2, applying $j_{\text{rs}!*}$ we get decompositions into indecomposable summands, and the indecomposable summand $j_{\text{rs}!*}(\underline{\mathbb{F}P}_F)$ has top and socle isomorphic to $j_{\text{rs}!*}\underline{F}$.

$$\begin{aligned}\mathcal{K} &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} (j_{\text{rs}!*}\underline{P}_F)^{\dim F} \\ \mathbb{F}\mathcal{K} &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} (j_{\text{rs}!*}\underline{\mathbb{F}P}_F)^{\dim F}\end{aligned}$$

The morphism $\pi_{\mathcal{N}}$ is proper and semi-small, hence $\mathbb{E}\mathcal{K}_{\mathcal{N}}$ is perverse. The functor $i_{\mathcal{N}}^*(-)[-r]$ induces a morphism

$$(4.1) \quad \text{res} : \text{End}(\mathbb{E}\mathcal{K}) \longrightarrow \text{End}(\mathbb{E}\mathcal{K}_{\mathcal{N}})$$

4.2.2. *Springer correspondence by restriction.*

Theorem 4.1. *The morphism res in (4.1) is an isomorphism.*

In the $\mathbb{E} = \mathbb{K}$ case this was proved by Borho and MacPherson in [BM81]. They argued as follows. The Weyl group W acts on G/T by $gT \cdot w = gn_wT$, where n_w is any representative of w in $N_G(T)$. So W acts naturally on the cohomology complex $\text{R}\Gamma(G/T, \mathbb{E})$. Since the projection $G/T \rightarrow G/B$ is a locally trivial U -fibration, it induces an isomorphism $\text{R}\Gamma(G/B, \mathbb{E}) \simeq \text{R}\Gamma(G/T, \mathbb{E})$, and thus there is a natural action of W on $\text{R}\Gamma(G/B, \mathbb{E})$, and hence on the cohomology $H^*(G/B, \mathbb{E})$. Let us call it the classical action. On the other hand, the stalk at 0 of \mathcal{K} is isomorphic to $\text{R}\Gamma(G/B, \mathbb{E})$, and $\mathbb{E}W$ acts on it through $\mathbb{E}W \simeq \text{End}(\mathcal{K}) \xrightarrow{\text{res}} \text{End}(\mathcal{K}_{\mathcal{N}})$. Let us call it the Lusztig action [Lus81]. In fact, the two actions coincide. When $\mathbb{E} = \mathbb{K}$, one can show that the classical action on the cohomology is the regular representation. Since the regular representation is faithful, this implies that the morphism res is injective. Then one can show that the two algebras have the same dimension using results from [Ste76]. This proves that res is an isomorphism in the case $\mathbb{E} = \mathbb{K}$.

We have

$$\mathbb{K}\mathcal{K} = \bigoplus_{E \in \text{Irr } \mathbb{K}W} j_{\text{rs}!*}\underline{E}^{\dim E}$$

and $i_{\mathcal{N}}^*(\mathbb{K}\mathcal{K})[-r] = \mathbb{K}\mathcal{K}_{\mathcal{N}}$. In fact, the restriction functor $i_{\mathcal{N}}^*[-r]$ sends each simple constituent $j_{\text{rs}!*}\underline{E}$ on a simple object. The assignment

$$\mathcal{R} : E \mapsto i_{\mathcal{N}}^*j_{\text{rs}!*}\underline{E}[-r]$$

is an injective map from $\text{Irr } \mathbb{K}W$ to the simple G -equivariant perverse sheaves on \mathcal{N} , which are parametrized by the pairs $(\mathcal{O}, \mathcal{L})$, where \mathcal{O} is a nilpotent orbit, and \mathcal{L} is an irreducible G -equivariant \mathbb{K} -local system on \mathcal{O} . This is the Springer correspondence (by restriction).

For example, for $G = SL_n$, the Specht module S^λ is sent to $\mathcal{IC}(\overline{\mathcal{O}}_\lambda, \mathbb{K})$, where \mathcal{O}_λ is the nilpotent orbit corresponding to the partition λ by the Jordan normal form.

When $\mathbb{E} = \mathbb{O}$ or \mathbb{F} , several difficulties arise with the Borho-MacPherson approach to prove Theorem 4.1. First, if we consider the cohomology of the stalk at 0, then one loses information, and obtains a non-faithful representation of the Weyl group, already in the case of $G = SL_2$. It is likely that the map from \mathbb{E} to the endomorphism algebra of the stalk at 0 considered as a (perfect) complex of $\mathbb{E}W$ -modules is injective. This would be one way to prove the injectivity, but this complex does not seem so easy to understand. Also, working over \mathbb{O} , one cannot invoke a dimension argument to conclude that we have an isomorphism.

To avoid these problems, we will use another classical approach to the Springer theory, using a Fourier transform. This gives a different action of the Weyl group on the Springer sheaf. Actually both actions differ by the sign character of the Weyl group, as was proved by Hotta in [Hot81] with \mathbb{K} coefficients, and in [AHJR14b] with general \mathbb{E} coefficients. So we can deduce Theorem 4.1 from the corresponding result about the Fourier approach.

We note that Riche has been able to prove Theorem 4.1 directly, using the Ginzburg interpretation of the endomorphism algebra of the Springer sheaf in terms of the Borel-Moore homology of the Steinberg variety [Ric13].

4.2.3. The Fourier-Deligne transform of $\mathbb{E}\mathcal{K}$. We assume that there exists a non-degenerate G -invariant symmetric bilinear form μ on \mathfrak{g} , so that we can identify \mathfrak{g} with its dual. This is the case, for example, if p is very good for G (take the Killing form), or if $G = GL_n$ (take $\mu(X, Y) = \text{tr}(XY)$). For a more detailed discussion, see [Let05].

Lemma 4.2. *The root subspace \mathfrak{g}_α is orthogonal to \mathfrak{t} and to all the root subspaces \mathfrak{g}_β with $\beta \neq -\alpha$.*

Proof. Let $x \in \mathfrak{t}$. For $t \in T$, we have $\mu(x, e_\alpha) = \mu(\text{Ad}(t)x, \text{Ad}(t)e_\alpha) = \alpha(t)\mu(x, e_\alpha)$. Since $\alpha \neq 0$, we can choose t so that $\alpha(t) \neq 1$, and thus $\mu(x, e_\alpha) = 0$.

Now let β be a root different from $-\alpha$. We have $\mu(e_\beta, e_\alpha) = \mu(\text{Ad}(t)e_\beta, \text{Ad}(t)e_\alpha) = \alpha(t)\beta(t)\mu(e_\beta, e_\alpha)$. Since $\beta \neq -\alpha$, we may choose t so that $\alpha(t)\beta(t) \neq 1$, and thus $\mu(e_\beta, e_\alpha) = 0$. \square

Corollary 4.3. *The orthogonal of \mathfrak{b} in \mathfrak{g} is \mathfrak{u} .*

Proof. By the preceding lemma, \mathfrak{b} is orthogonal to \mathfrak{u} , and we have $\dim \mathfrak{b} + \dim \mathfrak{u} = 2\nu + r = \dim \mathfrak{g}$, hence the result, since μ is non-degenerate. \square

Let \mathcal{F} be the Fourier-Deligne transform associated to $p : \mathfrak{g} \rightarrow S = \text{Spec } k$ (any vector space can be considered as a vector bundle over a point). Since we identify \mathfrak{g} with \mathfrak{g}' , the functor \mathcal{F} is an auto-equivalence of the triangulated category $D_c^b(\mathfrak{g}, \mathbb{E})$. The application a of Theorem 3.3 (INV), which was defined as the opposite of the canonical isomorphism from a vector bundle to its bidual, is now multiplication by -1 .

We will need to consider the base change $f : \mathcal{B} \rightarrow \text{Spec } k$. We will denote by $\mathcal{F}_{\mathcal{B}}$ the Fourier-Deligne transform associated to $p_{\mathcal{B}} : \mathcal{B} \times \mathfrak{g} \rightarrow \mathcal{B}$. We have a commutative

diagram

$$\begin{array}{ccccc}
G \times_B \mathfrak{g} & \longleftarrow & G \times_B \mathfrak{b} & \longleftarrow & G \times_B \mathfrak{u} \\
\parallel & & \parallel & & \parallel \\
\mathcal{B} \times \mathfrak{g} & \xleftarrow{i} & \tilde{\mathfrak{g}} & \xleftarrow{i_{\tilde{\mathcal{N}}}} & \tilde{\mathcal{N}} \\
\downarrow p_{\mathcal{B}} & \searrow \Delta & \downarrow \pi & \square_{\mathcal{N}} & \downarrow \pi_{\mathcal{N}} \\
\mathcal{B} & \xrightarrow{F} & \mathfrak{g} & \xleftarrow{i_{\mathcal{N}}} & \mathcal{N} \\
& \searrow f & \downarrow p & & \\
& & \text{Spec } k & &
\end{array}$$

We have (below COM means isomorphisms coming from commutative diagrams)

$$\begin{aligned}
\mathcal{F}(\mathbb{E}\mathcal{K}) &= \mathcal{F}(\pi_! \mathbb{E}_{\tilde{\mathfrak{g}}}[2\nu + r]) \\
&= \mathcal{F}(F_! i_* \mathbb{E}_{\tilde{\mathfrak{g}}}[\nu + r])[\nu] && \text{by COM}_1(\Delta) \\
&= F_! \mathcal{F}_{\mathcal{B}}(i_* \mathbb{E}_{\tilde{\mathfrak{g}}}[\nu + r])[\nu] && \text{by BC}_1(f) \\
&= F_! i_* i_{\tilde{\mathcal{N}}}^* \mathbb{E}_{\tilde{\mathcal{N}}}(-\nu - r)[\nu][\nu] && \text{by SUB} \\
&= i_{\mathcal{N}*} \pi_{\mathcal{N}*} \mathbb{E}_{\tilde{\mathcal{N}}}(-\nu - r)[2\nu] && \text{by COM}_1(\Delta, \square_{\mathcal{N}}) \\
&= i_{\mathcal{N}*} \mathbb{E}_{\mathcal{K}_{\mathcal{N}}}(-\nu - r)
\end{aligned}$$

Applying \mathcal{F} and using Theorem 3.3 (INV), we get

$$a_* \mathbb{E}\mathcal{K}(-2\nu - r) = \mathcal{F}(i_{\mathcal{N}*} \mathbb{E}_{\mathcal{K}_{\mathcal{N}}})(-\nu - r)$$

But $a_* \mathbb{E}\mathcal{K} \simeq \mathbb{E}\mathcal{K}$ since $\mathbb{E}\mathcal{K}$ is monodromic (\mathbb{C}^* -equivariant), so that:

Theorem 4.4. *We have*

$$\begin{aligned}
\mathcal{F}(\mathbb{E}\mathcal{K}) &\simeq i_{\mathcal{N}*} \mathbb{E}_{\mathcal{K}_{\mathcal{N}}}(-\nu - r) \\
\mathcal{F}(i_{\mathcal{N}*} \mathbb{E}_{\mathcal{K}_{\mathcal{N}}}) &\simeq \mathbb{E}\mathcal{K}(-\nu)
\end{aligned}$$

Note that this proves a second time that $\mathcal{K}_{\mathcal{N}}$ is perverse.

Corollary 4.5. *The functors $j_{\text{rs}*}$, $\mathcal{F}(-)(\nu + r)$ and $i_{\mathcal{N}*}$ induce isomorphisms*

$$\mathbb{E}W = \text{End}(\mathbb{E}\mathcal{K}_{\text{rs}}) \xrightarrow{\sim} \text{End}(\mathbb{E}\mathcal{K}) \xrightarrow{\sim} \text{End}(i_{\mathcal{N}*} \mathbb{E}_{\mathcal{K}_{\mathcal{N}}}) \xleftarrow{\sim} \text{End}(\mathbb{E}_{\mathcal{K}_{\mathcal{N}}})$$

For $E \in \mathbb{E}W\text{-mod}$, let $\mathcal{T}(E) = \mathcal{F}j_{\text{rs}*}(E[2\nu + r])(\nu + r)$. By the theorem above, we have $\mathcal{T}(\mathbb{E}W) = \mathcal{K}_{\mathcal{N}}$. More correctly, we should say that $\mathcal{T}(\mathbb{E}W)$ is supported on \mathcal{N} , and write $\mathcal{T}(\mathbb{E}W) = i_{\mathcal{N}*} \mathcal{K}_{\mathcal{N}}$, but we identify the perverse sheaves on \mathcal{N} with their extension by zero on \mathfrak{g} .

Corollary 4.6. *The perverse sheaf $\mathbb{K}\mathcal{K}_{\mathcal{N}}$ is semisimple, and we have the decomposition*

$$\mathbb{K}\mathcal{K}_{\mathcal{N}} = \bigoplus_{E \in \text{Irr } \mathbb{K}W} \mathcal{T}(E)^{\dim E}.$$

Similarly, we have decompositions into indecomposable summands

$$\mathcal{K}_{\mathcal{N}} = \bigoplus_{F \in \text{Irr } \mathbb{F}W} \mathcal{T}(P_F)^{\dim F},$$

and

$$\mathbb{F}\mathcal{K}_{\mathcal{N}} = \bigoplus_{F \in \text{Irr } \mathbb{F}W} \mathcal{T}(\mathbb{F}P_F)^{\dim F}.$$

The indecomposable summand $\mathcal{T}(\mathbb{F}P_F)$ has top and socle isomorphic to $\mathcal{T}(F)$.

4.2.4. *Springer correspondence by Fourier-Deligne transform.* We are now ready to define a Springer correspondence by Fourier-Deligne transform. We treat the ordinary and modular cases simultaneously. The ordinary case was achieved by Brylinski [Bry86], following the case of \mathcal{D} -modules over complex varieties [HK84]. See the survey by Shoji [Sho88] for a nice exposition.

At the time of [Jut07], the modular case was new. All the steps go through as in the ordinary case, except the fact that $\mathbb{F}\mathcal{K}_{\mathcal{N}}$ is no longer semisimple in general. However, to each simple $\mathbb{F}W$ -module, we can still associate a simple G -equivariant perverse sheaf on the nilpotent cone. Recall that we denote by $\mathfrak{P}_{\mathbb{E}}$ the set of pairs $(\mathcal{O}, \mathcal{L})$ where \mathcal{O} is a nilpotent orbit and \mathcal{L} is a G -equivariant irreducible local system on \mathcal{O} with coefficients in $\mathbb{E} = \mathbb{K}$ or \mathbb{F} . We will often abuse notation and identify it with the set of pairs (x, ρ) with $x \in \mathcal{N}$ and $\rho \in \text{Irr } \mathbb{E}A_G(x)$, up to G -conjugacy. For $(x, \rho) \in \mathfrak{P}_{\mathbb{E}}$, we denote by $\mathcal{IC}(x, \rho)$ the corresponding simple G -equivariant perverse sheaf.

Theorem 4.7. *Let \mathbb{E} be \mathbb{K} or \mathbb{F} . For $E \in \text{Irr } \mathbb{E}W$, let*

$$\mathcal{T}_{\mathbb{E}}(E) = \mathcal{F}j_{\text{rs}!}(\underline{E})$$

Then $\mathcal{T}_{\mathbb{E}}(E)$ is a simple G -equivariant perverse sheaf supported by \mathcal{N} . Hence it is of the form $\mathcal{IC}_{\mathbb{E}}(x_E, \rho_E)$ for some $(x_E, \rho_E) \in \mathfrak{P}_{\mathbb{E}}$. The assignment $E \mapsto (x_E, \rho_E)$ defines an injective map

$$\Psi_{\mathbb{E}} : \text{Irr } \mathbb{E}W \longrightarrow \mathfrak{P}_{\mathbb{E}}.$$

Proof. Since E is an irreducible $\mathbb{E}W$ -module, the shifted local system \underline{E} is a simple perverse sheaf on \mathfrak{g}_{rs} , so its intermediate extension $j_{\text{rs}!}(\underline{E})$ is a simple perverse sheaf on \mathfrak{g} . Since the Fourier-Deligne transform maps a simple perverse sheaf to a simple perverse sheaf by Corollary 3.13, and preserves G -equivariance by Proposition 3.8, we conclude that $\mathcal{T}_{\mathbb{E}}(E)$ is a simple G -equivariant perverse sheaf.

Let us now prove that $\mathcal{T}_{\mathbb{E}}(E)$ is supported by \mathcal{N} . In the case $\mathbb{E} = \mathbb{K}$, one can simply say that $\mathcal{T}_{\mathbb{K}}(E)$ is a direct summand of $\mathbb{K}\mathcal{K}_{\mathcal{N}}$ by Corollary 4.6. In the case $\mathbb{E} = \mathbb{F}$, this is no longer true when ℓ divides the order of the Weyl group W . However, we have seen in Corollary 4.6 that $\mathcal{T}_{\mathbb{F}}(E)$ is isomorphic to the top and to the socle of $\mathcal{T}_{\mathbb{F}}(\mathbb{F}P_E)$, which is a direct summand of $\mathbb{F}\mathcal{K}_{\mathcal{N}}$. Thus $\mathcal{T}_{\mathbb{F}}(E)$ is supported by \mathcal{N} in the modular case as well.

It follows that $\mathcal{T}_{\mathbb{E}}(E)$ is of the form $\mathcal{IC}_{\mathbb{E}}(x_E, \rho_E)$ for some uniquely determined $(x_E, \rho_E) =: \Psi_{\mathbb{E}}(E)$ in $\mathfrak{P}_{\mathbb{E}}$. The map $\Psi_{\mathbb{E}}$ is injective, because, if $\Psi_{\mathbb{E}}(E) = (x_E, \rho_E)$, then $\mathcal{T}_{\mathbb{E}}(E) = \mathcal{IC}_{\mathbb{E}}(x_E, \rho_E)$, and E is the simple $\mathbb{E}W$ -module corresponding to the local system $j_{\text{rs}}^* \mathcal{F}^{-1} \mathcal{IC}_{\mathbb{E}}(x_E, \rho_E)$. We have used the inverse Fourier transform \mathcal{F}^{-1} of Theorem 3.3. \square

Definition 4.8. The map $\Psi_{\mathbb{E}}$ of Theorem 4.7 is the Springer correspondence for $\mathbb{E}W$ (by Fourier-Deligne transform).

So the Springer correspondence $\Psi_{\mathbb{E}}$ induces a bijection from $\text{Irr } \mathbb{E}W$ onto its image. In the ordinary case, this image contains all pairs involving a trivial local system. This is no longer true in the modular case. However, the following proposition shows that the trivial pair is always in the image.

Proposition 4.9. *Let \mathbb{E} be \mathbb{K} or \mathbb{F} . The Springer correspondence for $\mathbb{E}W$ maps the trivial $\mathbb{E}W$ -module to the trivial nilpotent pair. That is, if 1_W denotes the trivial $\mathbb{E}W$ -module, then we have*

$$\Psi_{\mathbb{E}}(1_W) = (0, 1).$$

Proof. We have $\mathcal{T}_{\mathbb{E}}(1_W) = \mathcal{F}(j_{rs!} \mathbb{E}_{\mathbb{E}_{rs}}) = \mathcal{F}(\mathbb{E}_{\mathbb{E}}) = \mathcal{IC}_{\mathbb{E}}(0, 1)$ by Proposition 3.2 (DIRAC). \square

Recall that the Springer correspondence by Fourier transform differs from the Springer correspondence by restriction. For example, we have

$$\mathcal{R}_{\mathbb{K}}(1_W) = i_{\mathcal{N}}^* j_{rs!} \mathbb{K}_{\mathbb{E}_{rs}}[-r] = i_{\mathcal{N}}^* \mathbb{K}_{\mathbb{E}}[2\nu] = \mathbb{K}_{\mathcal{N}}[2\nu] = \mathcal{IC}_{\mathbb{K}}(x_{\text{reg}}, 1)$$

(see the remarks at the end of 4.2.2).

We note that $\mathcal{T}_{\mathbb{E}}$ induces an equivalence of categories from $\mathbb{E}W\text{-mod}$ onto its essential image, since it is fully faithful. However, the inclusion of this image into the category of G -equivariant \mathbb{E} -perverse sheaves on \mathcal{N} (as a full subcategory) does not preserve Ext groups.

Let us remark that if ℓ does not divide the order of any of the finite groups $A_G(x)$, $x \in \mathcal{N}$, then all the group algebras $\mathbb{F}A_G(x)$ are semisimple, so for each x there is a natural bijection $\text{Irr } \mathbb{K}A_G(x) \xrightarrow{\sim} \text{Irr } \mathbb{F}A_G(x)$, and thus there is a natural bijection $\mathfrak{P}_{\mathbb{K}} \xrightarrow{\sim} \mathfrak{P}_{\mathbb{F}}$.

5. DECOMPOSITION MATRICES AND BASIC SETS

We have decomposition maps for representations of the Weyl group W and for perverse sheaves on the nilpotent cone \mathcal{N} . In this section, we are going to compare them. The main result is that the decomposition matrix of a Weyl group is a submatrix of the decomposition matrix for G -equivariant perverse sheaves on \mathcal{N} .

Let us recall some results of Subsection 2.1 in the context of the nilpotent cone. We have the decomposition matrix $D^{\mathcal{N}}$ for G -equivariant perverse sheaves on the nilpotent cone: for $(x, \rho) \in \mathfrak{P}_{\mathbb{K}}$ and $(y, \sigma) \in \mathfrak{P}_{\mathbb{F}}$, recall that

$$d_{(x, \rho), (y, \sigma)}^{\mathcal{N}} := [\mathbb{F} \mathcal{IC}(\overline{\mathcal{O}}_x, \mathcal{L}_{\rho, \mathbb{O}}) : \mathcal{IC}(\overline{\mathcal{O}}_y, \mathcal{L}_{\sigma})],$$

where \mathcal{O}_x (resp. \mathcal{O}_y) is the orbit of x (resp. y), and \mathcal{L}_{ρ} (resp. \mathcal{L}_{σ}) is the local system associated to ρ (resp. σ). Recall that this number does not depend on the choice of an integral form $\mathcal{L}_{\rho, \mathbb{O}}$. Those decomposition numbers satisfy the following block triangularity properties:

$$(5.1) \quad d_{(x, \rho), (x, \sigma)}^{\mathcal{N}} = d_{\rho, \sigma}^{A_G(x)};$$

$$(5.2) \quad d_{(x, \rho), (y, \sigma)}^{\mathcal{N}} \neq 0 \implies \mathcal{O}_y \leq \mathcal{O}_x.$$

Actually, the decomposition map d is only one side of the Brauer-Nesbitt *cde*-triangle. We will first see that the e coefficients can be interpreted as multiplicities of simple perverse sheaves in the perverse sheaves $\mathcal{T}(\mathbb{K}P_F)$ (we do not claim that these perverse sheaves are projective, however).

After that, we will see that the Springer correspondence allows us to define *Springer basic sets* for Weyl groups. This gives a geometric proof that the decomposition matrices of Weyl groups are unitriangular. Geck and Rouquier have defined *canonical basic sets* for Hecke algebras, under some assumptions on the characteristic, using Lusztig's a function [GR01]. It would be interesting to compare both approaches.

5.1. Comparison of e maps.

Theorem 5.1. *Let $F \in \text{Irr } \mathbb{F}W$. Then $\mathcal{T}(\mathbb{K}P_F)$ is supported on \mathcal{N} , and for each $E \in \text{Irr } \mathbb{K}W$ we have*

$$[\mathbb{K}P_F : E] = [\mathcal{T}(\mathbb{K}P_F) : \mathcal{T}(E)].$$

Proof. We have

$$\bigoplus_{F \in \mathbb{F}W} \mathcal{T}(\mathbb{K}P_F)^{\dim F} = \mathcal{T}(\mathbb{K}W) = \mathbb{K}\mathcal{K}_{\mathcal{N}}$$

hence $\mathcal{T}(\mathbb{K}P_F)$ is supported on \mathcal{N} . Moreover,

$$\begin{aligned} [\mathbb{K}P_F : E] &= [j_{rs!} \mathbb{K}P_F : j_{rs!} E] && \text{by Corollary 2.4} \\ &= [\mathcal{F}j_{rs!} \mathbb{K}P_F(\nu + r) : \mathcal{F}j_{rs!} E(\nu + r)] \\ &= [\mathbb{K}\mathcal{T}(P_F) : \mathcal{T}(E)] \end{aligned}$$

□

5.2. Comparison of d maps. If $E \in \text{Irr } \mathbb{K}W$ and $F \in \text{Irr } \mathbb{F}W$, let $d_{E,F}^W$ be the corresponding decomposition number.

Theorem 5.2. *Let $E \in \text{Irr } \mathbb{K}W$, and let $E_{\mathbb{O}}$ be an integral form for E . Then $\mathcal{T}(E_{\mathbb{O}})$ is supported on \mathcal{N} , and for each $F \in \text{Irr } \mathbb{F}W$ we have*

$$[\mathbb{F}E_{\mathbb{O}} : F] = [\mathbb{F}\mathcal{T}(E_{\mathbb{O}}) : \mathcal{T}(F)]$$

Thus

$$d_{E,F}^W = d_{\Psi_{\mathbb{K}}(E), \Psi_{\mathbb{F}}(F)}^{\mathcal{N}}$$

Proof. By [Jut09, (2.57)], we have a short exact sequence

$$(5.3) \quad 0 \longrightarrow T \longrightarrow \mathbb{F}j_{rs!} E_{\mathbb{O}} \longrightarrow j_{rs!} \mathbb{F}E_{\mathbb{O}} \longrightarrow 0$$

with T supported on $\mathfrak{g} - \mathfrak{g}_{rs}$.

We have $\mathbb{K}\mathcal{T}(E_{\mathbb{O}}) = \mathcal{T}(E)$, so it is supported by \mathcal{N} . Now $\mathcal{T}(E_{\mathbb{O}})$ is torsion-free because $E_{\mathbb{O}}$ is torsion-free, $j_{rs!}$ preserves monomorphisms, and \mathcal{F} is an equivalence. Hence by Proposition 2.8 it is also supported by \mathcal{N} . Moreover,

$$\begin{aligned} &[\mathbb{F}E_{\mathbb{O}} : F] \\ &= [j_{rs!} \mathbb{F}E_{\mathbb{O}} : j_{rs!} F] && \text{by Corollary 2.5} \\ &= [\mathbb{F}j_{rs!} E_{\mathbb{O}} : j_{rs!} F] && \text{by (5.3) and } [T : j_{rs!} F] = 0 \\ &= [\mathbb{F}\mathcal{F}(j_{rs!} E_{\mathbb{O}})(\nu + r) : \mathcal{F}(j_{rs!} F)(\nu + r)] && \text{by EQUIV and } \mathcal{F}\mathbb{F} = \mathbb{F}\mathcal{F} \\ &= [\mathbb{F}\mathcal{T}(E_{\mathbb{O}}) : \mathcal{T}(F)]. \end{aligned}$$

□

This theorem means that we can obtain the decomposition matrix of the Weyl group W by extracting certain rows (the image $\mathfrak{P}_{\mathbb{K}}^0$ of the ordinary Springer correspondence) and certain columns (the image $\mathfrak{P}_{\mathbb{F}}^0$ of the modular Springer correspondence) of the decomposition matrix for G -equivariant perverse sheaves on the nilpotent cone \mathcal{N} .

5.3. Springer basic sets. Recall that an ℓ -modular basic set of a finite group is a set of Brauer characters which is a basis of the Grothendieck group of modular representations. Such a basic set is said to be ordinary if those Brauer characters are restrictions of ordinary characters to the set of ℓ -regular elements. Then this set of ordinary characters is usually called a basic set itself. Typically, one looks for ordinary basic sets such that the corresponding submatrix of the decomposition matrix is unitriangular (for a certain order on ordinary characters). Such a “triangular basic set” allows one to define an injection of Brauer characters to ordinary characters.

For a nilpotent orbit \mathcal{O} , let us denote by $A_G(\mathcal{O})$ the group $A_G(x_{\mathcal{O}})$ for some $x_{\mathcal{O}} \in \mathcal{O}$. Since we assume that G is simple adjoint, each group $A_G(\mathcal{O})$ is either

\mathfrak{S}_2^k , \mathfrak{S}_3 , \mathfrak{S}_4 or \mathfrak{S}_5 . In the case of a symmetric group, the ordinary irreducible representations are parametrized by partitions, they are naturally ordered by the dominance order, and we have a well-known triangular basic set given by the ℓ -regular partitions. In the case of \mathfrak{S}_2^k , we take the k -fold cartesian product of the ordered set $(2) > (1^2)$. In the case $\ell = 2$, we take the obvious basic set consisting of the trivial representation (corresponding to the choice of (2) in each factor). For each nilpotent orbit \mathcal{O} , let $\beta_{\mathcal{O}} : \text{Irr } \mathbb{F}A_G(\mathcal{O}) \hookrightarrow \text{Irr } \mathbb{K}A_G(\mathcal{O})$ denote the injections corresponding to the basic sets mentioned above. We denote by $\beta_{\mathcal{N}} : \mathfrak{P}_{\mathbb{F}} \hookrightarrow \mathfrak{P}_{\mathbb{K}}$ the injection obtained by taking the disjoint union of all $\beta_{\mathcal{O}}$. Recall that, if ℓ does not divide the orders of the $A_G(\mathcal{O})$, then this is actually a bijection.

In the following diagram, we would like to fill in the dashed line to obtain a *Springer basic set* given by the injection β_S .

Theorem 5.3. *There exists an injection $\beta_S : \text{Irr } \mathbb{F}W \hookrightarrow \text{Irr } \mathbb{K}W$ making the following diagram commutative:*

$$\begin{array}{ccc} \text{Irr } \mathbb{F}W & \xrightarrow{\Psi_{\mathbb{F}}} & \mathfrak{P}_{\mathbb{F}} \\ \downarrow \beta_S & & \downarrow \beta_{\mathcal{N}} \\ \text{Irr } \mathbb{K}W & \xrightarrow{\Psi_{\mathbb{K}}} & \mathfrak{P}_{\mathbb{K}} \end{array}$$

It has the following properties (unitriangularity):

$$(5.4) \quad \forall F \in \text{Irr } \mathbb{F}W, \quad d_{\beta_S(F), F}^W = 1$$

$$(5.5) \quad \forall E \in \text{Irr } \mathbb{K}W, \quad \forall F \in \text{Irr } \mathbb{F}W, \quad d_{E, F}^W \neq 0 \implies E \leq F$$

The first part can be reformulated as: each pair in the image of the modular Springer correspondence goes under $\beta_{\mathcal{N}}$ to a pair which is in the image of the ordinary Springer correspondence. The second part says that we can arrange the rows and columns of the decomposition matrix of W so that it is unitriangular, using the order on ordinary characters induced by the order on nilpotent pairs described above, and the map β_S . In other words, we have constructed a triangular basic set using the Springer correspondence. We call it the *Springer basic set*. To achieve this aim, we first need the following result.

Proposition 5.4. *For $(x, \rho) \in \mathfrak{P}_{\mathbb{K}}$ and $(y, \sigma) \in \mathfrak{P}_{\mathbb{F}}$, if $(x, \rho) \notin \text{Im } \Psi_{\mathbb{K}}$ and $d_{(x, \rho), (y, \sigma)}^{\mathcal{N}} \neq 0$ then $(y, \sigma) \notin \text{Im } \Psi_{\mathbb{F}}$.*

Proof. Since $(x, \rho) \notin \text{Im } \Psi_{\mathbb{K}}$, we have $j_{\text{rs}}^* \mathcal{F}^{-1} \mathcal{IC}_{\mathbb{K}}(x, \rho) = 0$. In other words, we are in the situation of Proposition 2.8 with $U = \mathfrak{g}_{\text{rs}}$ and $Z = \mathfrak{g} \setminus \mathfrak{g}_{\text{rs}}$, and $\mathcal{F}^{-1} \mathcal{IC}_{\mathbb{K}}(x, \rho)$ is supported by Z . We deduce that its integral form $\mathcal{F}^{-1} \mathcal{IC}_{\mathbb{O}}(x, \rho_{\mathbb{O}})$ (for any choice of $\rho_{\mathbb{O}}$) is also supported by Z , and also the modular reduction $\mathbb{F} \mathcal{F}^{-1} \mathcal{IC}_{\mathbb{O}}(x, \rho_{\mathbb{O}})$. Thus we have the following equality in the Grothendieck group of \mathbb{F} -local systems on \mathfrak{g}_{rs} :

$$\begin{aligned} [j_{\text{rs}}^* \mathbb{F} \mathcal{F}^{-1} \mathcal{IC}_{\mathbb{O}}(x, \rho_{\mathbb{O}})] &= [j_{\text{rs}}^* \mathcal{F}^{-1} \mathbb{F} \mathcal{IC}_{\mathbb{O}}(x, \rho_{\mathbb{O}})] \\ &= \sum_{(y, \sigma) \in \mathfrak{P}_{\mathbb{F}}} d_{(x, \rho), (y, \sigma)}^{\mathcal{N}} [j_{\text{rs}}^* \mathcal{F}^{-1} \mathcal{IC}_{\mathbb{F}}(y, \sigma)] = 0. \end{aligned}$$

The $[j_{\text{rs}}^* \mathcal{F}^{-1} \mathcal{IC}_{\mathbb{F}}(y, \sigma)]$ which are non-zero are the classes of distinct simple objects, hence are linearly independent. Now if $(y, \sigma) = \Psi_{\mathbb{F}}(F)$, then $j_{\text{rs}}^* \mathcal{F}^{-1} \mathcal{IC}_{\mathbb{F}}(y, \sigma) = \underline{E}[2\nu + r] \neq 0$. We can conclude that $d_{(x, \rho), (y, \sigma)}^{\mathcal{N}} = 0$ for all $(y, \sigma) \in \text{Im } \Psi_{\mathbb{F}}$. \square

Proof of theorem. Let $F \in \text{Irr } \mathbb{F}W$. We want to prove that $\beta_{\mathcal{N}}(\Psi_{\mathbb{F}}(F))$ is in $\text{Im } \Psi_{\mathbb{K}}$. Let $(x, \sigma) := \Psi_{\mathbb{F}}(F)$, and $\rho := \beta_x(\sigma)$ so that $(x, \rho) = \beta_{\mathcal{N}}(x, \sigma)$. Then $d_{(x, \rho), (x, \sigma)}^{\mathcal{N}} = 1 \neq 0$, hence, by Proposition 5.4, $(x, \rho) = \Psi_{\mathbb{K}}(E)$ for some $E \in \text{Irr } \mathbb{K}W$ (which is uniquely determined since $\Psi_{\mathbb{K}}$ is injective). We define $\beta_S(F) := E$. The diagram commutes by construction. Moreover, for $F \in \text{Irr } \mathbb{F}W$ with $\Psi_{\mathbb{F}}(F) = (y, \sigma)$, we have

$$d_{\beta_S(F), F}^W = d_{\Psi_{\mathbb{K}}(\beta_S(F)), \Psi_{\mathbb{F}}(F)}^{\mathcal{N}} = d_{\beta_{\mathcal{N}}(\Psi_{\mathbb{F}}(F)), \Psi_{\mathbb{F}}(F)}^{\mathcal{N}} = d_{(y, \beta_y(\sigma)), (y, \sigma)}^{\mathcal{N}} = d_{\beta_y(\sigma), \sigma}^{A_G(y)} = 1,$$

and if in addition $E \in \text{Irr } \mathbb{E}W$ is such that $d_{E, F}^W \neq 0$, then let $(x, \rho) = \Psi_{\mathbb{K}}(E)$

$$d_{(x, \rho), (y, \sigma)}^{\mathcal{N}} = d_{\Psi_{\mathbb{K}}(E), \Psi_{\mathbb{F}}(F)}^{\mathcal{N}} = d_{E, F}^W \neq 0$$

hence $x \leq y$, and in case of equality we have $d_{\rho, \sigma}^{A_G(x)} \neq 0$, hence $\beta_x(\sigma) \leq \rho$. By definition, this means $E \leq \beta_S(F)$. \square

6. SOME DECOMPOSITION NUMBERS

We refer to [Jut09] for some methods to compute decomposition numbers for perverse sheaves. The calculations for the extreme cases (regular and subregular orbits, or minimal and trivial orbits) were done there. We assume that G is almost simple, so that there is a unique subregular nilpotent orbit and a unique minimal (non-zero) nilpotent orbit.

6.1. Subregular nilpotent orbit. The singularity of the nilpotent cone along the subregular orbit is a simple singularity [Bri71], meaning that the intersection of the nilpotent cone with a transverse slice to the subregular orbit is isomorphic to the quotient of \mathbb{A}^2 by a finite subgroup of SL_2 . Those singularities have an *ADE* classification. We will denote by x_{reg} (resp. x_{subreg}) a regular (resp. subregular) nilpotent element.

Let Γ be the type of \mathfrak{g} . If Γ is simply laced, then the nilpotent cone has a simple singularity of the same type Γ along the subregular orbit. The general case is explained in [Slo80b]. One can associate to Γ a simply-laced diagram $\widehat{\Gamma}$ and a subgroup A of its automorphism group, so that Γ can be seen as a kind of folding of $\widehat{\Gamma}$. We assume that G is adjoint. Then this group A can be identified with $A_G(x_{\text{subreg}})$. Let $\widehat{\Phi}$ be a root system of type $\widehat{\Gamma}$, with weight lattice $P(\widehat{\Phi})$ and root lattice $Q(\widehat{\Phi})$. The group A acts naturally on $P(\widehat{\Phi})/Q(\widehat{\Phi})$. Then the singularity of the nilpotent cone along the subregular orbit, along with the symmetries induced by $A_G(x_{\text{subreg}})$, is a simple singularity of type $\widehat{\Gamma}$, with the action of A by automorphisms.

Using this geometric description, the decomposition numbers involving the regular and subregular orbits have been computed in [Jut09, Section 4]. The result is the following:

Theorem 6.1. *We assume that G is adjoint. Then the decomposition numbers for the regular and subregular orbits are given by*

$$d_{(x_{\text{reg}}, 1), (x_{\text{subreg}}, \rho)}^{\mathcal{N}} = \left[\mathbb{F} \otimes_{\mathbb{Z}} \left(P(\widehat{\Phi})/Q(\widehat{\Phi}) \right) : \rho \right]$$

where ρ runs over the irreducible representations of $\mathbb{F}A_G(x_{\text{subreg}})$.

6.2. Minimal nilpotent orbit. We denote the minimal nilpotent orbit of \mathfrak{g} by \mathcal{O}_{\min} . Its closure is $\overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \mathcal{O}_{\text{triv}}$, where $\mathcal{O}_{\text{triv}}$ is the trivial orbit $\{0\}$. It is of dimension $2h^\vee - 2$, where h^\vee is the dual Coxeter number.

Remember that we denote by Φ the root system of \mathfrak{g} with respect to some Cartan subalgebra \mathfrak{t} . Now we choose some basis of Φ , and we denote by Φ' the root subsystem generated by the long simple roots. Let W' be the Weyl group of Φ' . We have an epimorphism $W \rightarrow W'$. Let $\text{nat}_{W'}$ denote the reflection representation of W' . We denote by χ_{\min} its lift to W . Then we have $\Psi_{\mathbb{K}}(\chi_{\min}) = (x_{\min}, 1)$. Note that, if Φ is of simply-laced type, then $\Phi' = \Phi$, $W' = W$, $\chi_{\min} = \text{nat}_W$. On the other hand, we know that $\Psi_{\mathbb{F}}(1_W) = (0, 1)$, and of course the trivial representation 1_W is the lift to W of the trivial representation $1_{W'}$ of W' . By Theorem 5.2 and [Jut09, Section 5], we have the following result:

Theorem 6.2. *The decomposition numbers for the minimal and trivial orbits are given by:*

$$d_{\chi_{\min}, 1_W}^W = d_{\text{nat}_{W'}, 1_{W'}}^{W'} = d_{(x_{\min}, 1), (0, 1)}^{\mathcal{N}} = \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} (P^\vee(\Phi')/Q^\vee(\Phi')).$$

6.3. Special nilpotent orbits. In [Lus79], Lusztig introduced the special representations of a finite Weyl group. The *special nilpotent orbits* of \mathfrak{g} are the nilpotent orbits \mathcal{O} such that the representation of $\mathbb{K}W$ corresponding (via restriction) to the pair $(\mathcal{O}, \mathbb{K})$ is *special*. (The original definitions were in terms of unipotent classes but of course one can work with nilpotent orbits instead.)

On the other hand, Spaltenstein introduced an order-reversing map d from the set of nilpotent orbits to itself, such that $d^3 = d$ (it is an involution on its image) in [Spa78]. The image of d consists exactly of the special nilpotent orbits, and the locally closed subvarieties

$$\widehat{\mathcal{O}} = \overline{\mathcal{O}} \setminus \bigcup_{\substack{\mathcal{O}' \text{ special} \\ \overline{\mathcal{O}'} \subset \overline{\mathcal{O}}}} \overline{\mathcal{O}'}$$

where \mathcal{O} runs through the special orbits, form a partition of the nilpotent cone (any nilpotent orbit is contained in a $\widehat{\mathcal{O}}$ for a unique special orbit \mathcal{O}).

In type A , all the nilpotent orbits are special, so the special pieces are just the nilpotent orbits. Based on the particular case of the minimal special orbits, Lusztig conjectured in [Lus81] that special pieces are rationally smooth. This was confirmed in [BS84] for exceptional types, and in [KP89] for classical types. More precisely, Kraft and Procesi proved the following (they actually work with a complex group, but the result is certainly true for when the characteristic p of the base field is odd):

Theorem 6.3. *Let \mathcal{O} be a special nilpotent orbit of a classical group. Define the special piece $\widehat{\mathcal{O}}$ as above. Then $\widehat{\mathcal{O}}$ consists of 2^d conjugacy classes, where d is the number of irreducible components of $\widehat{\mathcal{O}} \setminus \mathcal{O}$. There is a smooth variety Y with an action of the group \mathfrak{S}_2^d , and an isomorphism*

$$(6.1) \quad Y/\mathfrak{S}_2^d \xrightarrow{\sim} \widehat{\mathcal{O}}$$

which identifies the stratification of $\widehat{\mathcal{O}}$ with the stratification of the quotient by isotropy groups. (These are the 2^d subproducts of \mathfrak{S}_2^d). In particular $\widehat{\mathcal{O}}$ is rationally smooth.

Actually their result gives more information: by Proposition 2.11, the variety $\widehat{\mathcal{O}}$ is not only \mathbb{K} -smooth, but also \mathbb{F} -smooth for $\ell \neq 2$. The following corollary is valid when the base field is \mathbb{C} , and probably over a base field of characteristic $p \neq 2$.

Corollary 6.4. *Let \mathcal{O} be a special nilpotent orbit in a classical group, with corresponding special piece $\widehat{\mathcal{O}}$. If $\ell \neq 2$ then $\widehat{\mathcal{O}}$ is \mathbb{F} -smooth and, for all orbits $\mathcal{O}' \subset \widehat{\mathcal{O}} \setminus \mathcal{O}$, and all \mathbb{F} -local systems \mathcal{L} on \mathcal{O}' , we have*

$$d_{(\mathcal{O}, \mathbb{F}), (\mathcal{O}', \mathcal{L})} = 0.$$

Seeing the result of Kraft and Procesi, Lusztig conjectured that also in exceptional types, any special piece $\widehat{\mathcal{O}}$ is the quotient of a smooth variety by a precise finite group $\Gamma_{\mathcal{O}}$ (which is a subgroup of Lusztig's canonical quotient of $A_G(\mathcal{O})$, see [Lus97]). In [FJLS] we prove a closely related result: if \mathcal{O}' denotes the minimal orbit in the special piece $\widehat{\mathcal{O}}$, and if V denotes the reflection representation of $\Gamma_{\mathcal{O}}$ (which is always a Coxeter group), then the intersection of $\widehat{\mathcal{O}}$ with a transversal slice to \mathcal{O}' is isomorphic to the quotient of $(V \oplus V^*)^k$ by $\Gamma_{\mathcal{O}}$, where k is some integer (usually one in exceptional types). Thus we can deduce a result like the above corollary for exceptional types, for the primes ℓ not dividing the order of $\Gamma_{\mathcal{O}}$.

7. THE CASE OF GL_n

In this section, we study the case of $G = GL_n$. Then $\mathfrak{g} = \mathfrak{gl}_n$ is the Lie algebra of all $n \times n$ matrices, and \mathcal{N} is the closed subset of nilpotent matrices in the usual sense, with GL_n acting by conjugation. Given a nilpotent matrix x in \mathcal{N} , the ordered list of the sizes of Jordan blocks in the Jordan normal form of x determines a partition of n . This defines a bijection between the nilpotent orbits and the set of partitions of n . We denote by \mathcal{O}_{λ} the orbit corresponding to λ , and by x_{λ} an element in \mathcal{O}_{λ} . We have $A_G(x_{\lambda}) = 1$ for all $\lambda \vdash n$, hence we will need to consider only constant local systems. So both $\mathfrak{P}_{\mathbb{K}}$ and $\mathfrak{P}_{\mathbb{F}}$ are in natural bijection with the set \mathfrak{P}_n of all partitions of n . To simplify the notation, we set $\mathcal{IC}_{\mathbb{E}}(\lambda) = \mathcal{IC}_{\mathbb{E}}(x_{\lambda}, 1)$ and $d_{\lambda, \mu}^{\mathcal{N}} := d_{(x_{\lambda}, 1), (x_{\mu}, 1)}^{\mathcal{N}}$.

7.1. Springer correspondence for GL_n . Let us recall the parametrization of simple modules for the symmetric group [Jam76]. The simple modules of $\mathbb{K}\mathfrak{S}_n$ are the Specht modules S^{λ} , where λ runs over the partitions of n . Those have an integral form $S_{\mathbb{O}}^{\lambda}$ equipped with a symmetric bilinear form β which is non-degenerate after extension of the scalars to \mathbb{K} . However the rank of those forms may drop after tensoring with \mathbb{F} . The module $D^{\mu} := \mathbb{F} \otimes_{\mathbb{O}} S_{\mathbb{O}}^{\mu} / \text{Rad}(\mathbb{F} \otimes_{\mathbb{O}} \beta)$ is non-zero exactly when μ is ℓ -regular (that is, all parts of μ have multiplicity strictly smaller than ℓ). The modules D^{μ} , for μ running over all ℓ -regular partitions of n , form a complete set of representatives of the isomorphism classes of $\mathbb{F}\mathfrak{S}_n$ -modules.

We note $d_{\lambda, \mu}^{\mathfrak{S}_n} := d_{S^{\lambda}, D^{\mu}}^{\mathfrak{S}_n}$. The decomposition matrix of \mathfrak{S}_n is unitriangular with respect to the dominance order of partitions: for λ and μ two partitions of n , where μ is ℓ -regular, we have

$$(7.1) \quad d_{\lambda, \lambda}^{\mathfrak{S}_n} = 1;$$

$$(7.2) \quad d_{\lambda, \mu}^{\mathfrak{S}_n} \neq 0 \implies \lambda \leq \mu.$$

Recall that we have a similar property for perverse sheaves (see (5.1)), again for the dominance order (now μ is not assumed to be ℓ -regular)

$$(7.3) \quad d_{\lambda,\lambda}^{\mathcal{N}} = 1;$$

$$(7.4) \quad d_{\lambda,\mu}^{\mathcal{N}} \neq 0 \implies \lambda \geq \mu.$$

In characteristic zero, it is known that $\Psi_{\mathbb{K}}$ is a bijection, which sends the Specht module S^λ to the pair $(x_{\lambda'}, 1)$, where λ' denotes the partition transposed to λ . We will see that those facts are enough to determine the modular Springer correspondence for $G = GL_n$.

Theorem 7.1. *Suppose $G = GL_n$. If μ is an ℓ -regular partition, then we have*

$$\Psi_{\mathbb{F}}(D^\mu) = (x_{\mu'}, 1)$$

where μ' is the partition transposed to μ . Thus we have, for two partitions λ and μ , with μ ℓ -regular, $d_{\lambda,\mu}^{\mathfrak{S}_n} = d_{\lambda',\mu'}^{\mathcal{N}}$.

Proof. Let μ be an ℓ -regular partition. We want to show that $\Psi_{\mathbb{F}}(\mu) = \mu'$. On the one hand, we have

$$d_{\mu',\Psi_{\mathbb{F}}(\mu)}^{\mathcal{N}} = d_{\Psi_{\mathbb{K}}(\mu),\Psi_{\mathbb{F}}(\mu)}^{\mathcal{N}} = d_{\mu,\mu}^{\mathfrak{S}_n} = 1 \neq 0$$

which proves that $\Psi_{\mathbb{F}}(\mu) \leq \mu'$.

On the other hand, we have

$$d_{\Psi_{\mathbb{F}}(\mu)',\mu}^{\mathfrak{S}_n} = d_{\Psi_{\mathbb{K}}(\Psi_{\mathbb{F}}(\mu)'),\Psi_{\mathbb{F}}(\mu)}^{\mathcal{N}} = d_{\Psi_{\mathbb{F}}(\mu),\Psi_{\mathbb{F}}(\mu)}^{\mathcal{N}} = 1 \neq 0$$

which proves that $\Psi_{\mathbb{F}}(\mu)' \leq \mu$, that is, $\Psi_{\mathbb{F}}(\mu) \geq \mu'$. \square

Note that this result has been generalized in [AHJR13].

7.2. Row and column removal rule. We introduce the “characteristic functions”

$$\begin{aligned} \chi_{\lambda,\mu} &= \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{K}} \mathcal{H}_{x_\mu}^i \mathcal{IC}(\overline{\mathcal{O}}_\lambda, \mathbb{K}) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{F}} \mathcal{H}_{x_\mu}^i \mathbb{F} \mathcal{IC}(\overline{\mathcal{O}}_\lambda, \mathbb{O}) \end{aligned}$$

and

$$\phi_{\nu,\mu} = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{F}} \mathcal{H}_{x_\mu}^i \mathcal{IC}(\overline{\mathcal{O}}_\nu, \mathbb{F})$$

These form triangular systems, in the sense that $\chi_{\lambda,\mu}$ can be non-zero only if $\mu \leq \lambda$, and $\phi_{\nu,\mu}$ can be non-zero only if $\mu \leq \nu$. We have

$$\chi_{\lambda,\mu} = \sum_{\nu} d_{\lambda,\nu} \phi_{\nu,\mu}$$

and $d_{\lambda,\nu}$ can be non-zero only if $\nu \leq \lambda$. Moreover, we have $\chi_{\lambda,\lambda} = \phi_{\lambda,\lambda} = 1$, and $d_{\lambda,\lambda} = 1$.

Kraft and Procesi found a row and column removal rule for the singularities of the closures of the nilpotent orbits in type A_{n-1} [KP81]. Actually, they state the result when the base field is \mathbb{C} . It is probably true also in characteristic p , but since this result is not available, for the rest of this section we work over \mathbb{C} . One can use the Fourier-Sato transform instead of the Fourier-Deligne transform and everything goes through.

Proposition 7.2. *Let $\mathcal{O}_\mu \subset \overline{\mathcal{O}}_\lambda$ be a degeneration of nilpotent orbits in \mathfrak{gl}_n and assume that the first r rows and the first s columns of λ and μ coincide. Denote by $\hat{\lambda}$ and $\hat{\mu}$ the Young diagrams obtained from λ and μ by erasing these rows and columns. Then $\mathcal{O}_{\hat{\mu}} \subset \overline{\mathcal{O}}_{\hat{\lambda}}$ in a smaller nilpotent cone $\hat{N} \subset \mathfrak{gl}_{\hat{n}}$, and we have*

$$\mathrm{codim}_{\overline{\mathcal{O}}_{\hat{\lambda}}} \mathcal{O}_{\hat{\mu}} = \mathrm{codim}_{\overline{\mathcal{O}}_\lambda} \mathcal{O}_\mu \quad \text{and} \quad \mathrm{Sing}(\overline{\mathcal{O}}_{\hat{\lambda}}, \mathcal{O}_{\hat{\mu}}) = \mathrm{Sing}(\overline{\mathcal{O}}_\lambda, \mathcal{O}_\mu)$$

All the partitions ν in the interval $[\mu, \lambda] = \{\nu \mid \mu \leq \nu \leq \lambda\}$ have the same first r rows and first s columns. For ν in $[\mu, \lambda]$, let us denote by $\hat{\nu}$ the partition obtained from ν by erasing them.

The proposition implies that, for all $\eta \leq \zeta$ in $[\mu, \lambda]$, the local intersection cohomology of $\overline{\mathcal{O}}_{\hat{\eta}}$ along $\mathcal{O}_{\hat{\zeta}}$ is the same as the local intersection cohomology of $\overline{\mathcal{O}}_\eta$ along \mathcal{O}_ζ , both over \mathbb{K} and over \mathbb{F} , and thus $\chi_{\eta, \zeta} = \chi_{\hat{\eta}, \hat{\zeta}}$ and $\phi_{\eta, \zeta} = \phi_{\hat{\eta}, \hat{\zeta}}$.

Since the decomposition numbers can be deduced from this information (for GL_n , we only have trivial $A_G(x)$), we find a row and column removal rule for the decomposition numbers for GL_n -equivariant perverse sheaves on the nilpotent cone of GL_n .

Proposition 7.3. *With the notations above, we have*

$$(7.5) \quad d_{\lambda, \mu}^N = d_{\hat{\lambda}, \hat{\mu}}^{\hat{N}}$$

Proof. The decomposition numbers $(d_{\eta, \zeta}^N)_{\mu \leq \zeta \leq \eta \leq \lambda}$ are the unique solution of the triangular linear system

$$\chi_{\eta, \zeta} = \sum_{\mu \leq \nu \leq \lambda} d_{\eta, \nu}^N \phi_{\nu, \zeta}$$

whereas the decomposition numbers $(d_{\hat{\eta}, \hat{\zeta}}^{\hat{N}})_{\hat{\mu} \leq \hat{\zeta} \leq \hat{\eta} \leq \hat{\lambda}}$ are the unique solution of the triangular linear system

$$\chi_{\hat{\eta}, \hat{\zeta}} = \sum_{\hat{\mu} \leq \hat{\nu} \leq \hat{\lambda}} d_{\hat{\eta}, \hat{\nu}}^{\hat{N}} \phi_{\hat{\nu}, \hat{\zeta}}$$

Since we have $\chi_{\eta, \zeta} = \chi_{\hat{\eta}, \hat{\zeta}}$ and $\phi_{\eta, \zeta} = \phi_{\hat{\eta}, \hat{\zeta}}$, the two systems are identical, so that $d_{\eta, \zeta}^N = d_{\hat{\eta}, \hat{\zeta}}^{\hat{N}}$ for all $\zeta \leq \eta$ in $[\mu, \lambda]$. In particular, we have $d_{\lambda, \mu}^N = d_{\hat{\lambda}, \hat{\mu}}^{\hat{N}}$. \square

8. THE GROTHENDIECK AND SPRINGER SHEAVES FOR SL_2

We will completely describe \mathcal{K} and \mathcal{K}_N in the case $G = SL_2$. Apart from the open stratum $\mathfrak{g}_{\mathrm{rs}}$, we just have the two nilpotent orbits $\mathcal{O}_{\mathrm{reg}} = \mathcal{O}_{\mathrm{min}} = \mathcal{O}_{(2)}$ and $\mathcal{O}_{\mathrm{triv}} = \mathcal{O}_{\mathrm{subreg}} = \mathcal{O}_{(1^2)} = \{0\}$. On $\mathfrak{g}_{\mathrm{rs}}$, we will only consider local systems which become trivial after a pullback by π_{rs} . We have $W = \mathfrak{S}_2$. The local system $\pi_{\mathrm{rs}*} \mathbb{E}$ corresponds to the regular representation $\mathbb{E}\mathfrak{S}_2$.

In characteristic 0, the group algebra is semi-simple, and the perverse sheaf $\mathbb{K}\mathcal{K}_{\mathrm{rs}}$ splits as the sum of the constant perverse sheaf C^{rs} and the shifted local system $C_\varepsilon^{\mathrm{rs}}$ corresponding to the sign representation of \mathfrak{S}_2 . These two simple components are sent by $j_{\mathrm{rs}*}$ on two simple perverse sheaves on \mathfrak{g} , the constant perverse sheaf C (since \mathfrak{g} is smooth), and the other one, C_ε . Let us denote by A the simple perverse sheaf supported on $\{0\}$, and by B the simple perverse sheaf $\mathcal{IC}(\overline{\mathcal{O}}_{\mathrm{reg}}, \mathbb{K})$. Since $\mathcal{F}(C) = A$, we must have $\mathcal{F}(C_\varepsilon) = B$. This gives the Springer correspondence for \mathfrak{sl}_2 by Fourier transform.

Let us make tables for the stalks of the perverse sheaves involved. We have a line for each stratum, and one column for each cohomology degree. If x is a point

of a given stratum \mathcal{O} and i is an integer, the corresponding entry in the table of a perverse sheaf \mathcal{A} will be the class of $\mathcal{H}_x^i \mathcal{A}$, seen as a representation of a suitable group $A(\mathcal{O})$, in the Grothendieck group of $\mathbb{E}A(\mathcal{O})$. There is a column χ describing the alternating sum of the stalks of each stratum.

Let us first describe $\mathbb{K}\mathcal{K}$. So, over \mathfrak{g}_{rs} , we have the regular representation of \mathfrak{S}_2 . Over \mathcal{O}_{reg} , the fibers are single points, so the cohomology of $\mathcal{B}_{x_{\text{reg}}}$ is just \mathbb{K} . But we have $\mathcal{B}_0 = \mathcal{B} = G/B = \mathbb{P}^1$. We get the following table for $\mathbb{K}\mathcal{K}$.

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	$-1 - \varepsilon$	$\mathbb{K} \oplus \mathbb{K}_\varepsilon$.	.	.
\mathcal{O}_{reg}	2	-1	\mathbb{K}	.	.	.
$\mathcal{O}_{\text{triv}}$	0	-2	\mathbb{K}	.	\mathbb{K}	.

It is the direct sum of the two simple perverse sheaves C

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	-1	\mathbb{K}	.	.	.
\mathcal{O}_{reg}	2	-1	\mathbb{K}	.	.	.
$\mathcal{O}_{\text{triv}}$	0	-1	\mathbb{K}	.	.	.

and C_ε , which we deduce by subtraction (we have a direct sum!)

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	$-\varepsilon$	\mathbb{K}_ε	.	.	.
\mathcal{O}_{reg}	2	0
$\mathcal{O}_{\text{triv}}$	0	-1	.	.	\mathbb{K}	.

The simple G -equivariant perverse sheaves on \mathcal{N} are $B = \mathcal{IC}(\overline{\mathcal{O}}_{\text{reg}}, \mathbb{K})$,

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	0
\mathcal{O}_{reg}	2	1	.	\mathbb{K}	.	.
$\mathcal{O}_{\text{triv}}$	0	1	.	\mathbb{K}	.	.

$A = \mathcal{IC}(\overline{\mathcal{O}}_{\text{triv}}, \mathbb{K})$

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	0
\mathcal{O}_{reg}	2	0
$\mathcal{O}_{\text{triv}}$	0	1	.	.	.	\mathbb{K}

and the cuspidal $B_\varepsilon = \mathcal{IC}(\overline{\mathcal{O}}_{\text{reg}}, \mathbb{K}_\varepsilon)$, which is clean (its intermediate extension is just the extension by zero), and stable by the Fourier-Deligne transform, by the general theory

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	0
\mathcal{O}_{reg}	2	1	.	\mathbb{K}_ε	.	.
$\mathcal{O}_{\text{triv}}$	0	1

We can check that, applying $i_{\mathcal{N}}^*[-1]$ to \mathcal{K} , we recover $\mathcal{K}_{\mathcal{N}}$. This functor sends C to B and C_ε to A . There is a twist by the sign character between the two versions of the Springer representations (by Fourier-Deligne transform, and by restriction).

So, to summarize the situation over \mathbb{K} , we have

\mathfrak{g}_{rs}	\mathfrak{g}	\mathcal{N}
$C^{\text{rs}} \oplus C_{\varepsilon}^{\text{rs}}$	$C \oplus C_{\varepsilon}$	$B \oplus A$

If $\ell \neq 2$, the situation over \mathbb{F} is similar. Now let us assume that $\ell = 2$. Then the sign representation becomes trivial. The regular representation is an extension of the trivial module by itself, so similarly $\mathbb{F}\mathcal{K}_{\text{rs}}$ is an extension of the constant c_{rs} (reduction of C_{rs}) by itself.

Now $\mathbb{F}\mathcal{K}$ is as follows:

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	-2	$\mathbb{F}\mathfrak{S}_2$.	.	.
\mathcal{O}_{reg}	2	-1	\mathbb{F}	.	.	.
$\mathcal{O}_{\text{triv}}$	0	-2	\mathbb{F}	.	\mathbb{F}	.

It must be made of the simple perverse sheaves a , b and c , where c is the constant on \mathfrak{g} (it is the reduction of C , and has the same table with \mathbb{F} instead of \mathbb{K}), a is the constant on the origin (the reduction of A), and $b = \mathcal{IC}(\overline{\mathcal{O}}_{\text{reg}}, \mathbb{F})$ has the following table

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	0
\mathcal{O}^{reg}	2	1	.	\mathbb{F}	.	.
$\mathcal{O}^{\text{triv}}$	0	0	.	\mathbb{F}	\mathbb{F}	.

Looking at the χ functions, we see that $[\mathbb{F}\mathcal{K}] = 2[c] + [b]$ in the Grothendieck group of ${}^p\mathcal{M}_G(\mathcal{N}, \mathbb{F})$. We know that the top and the socle of $\mathbb{F}\mathcal{K}$ must be c , the intermediate extension of c , and that b cannot appear either in the top nor in the socle. Thus there is only one possible Loewy structure:

$$\mathbb{F}\mathcal{K} = \begin{array}{c} c \\ b \\ c \end{array}$$

Similarly, we find

$$\mathbb{F}\mathcal{K}_{\mathcal{N}} = \begin{array}{c} a \\ b \\ a \end{array}$$

Thus, as we already know, $\mathcal{F}(c) = a$, but we also deduce that $\mathcal{F}(b) = b$.

The restriction functor $i_{\mathcal{N}}^*$ sends c to the reduction of B , which has the following Loewy structure (by Section 6 and the more detailed information in [Jut09]):

$$\begin{array}{c} b \\ a \end{array}$$

The reduction of C_{ε} has structure

$$\begin{array}{c} c \\ b \end{array}$$

and it restricts to a (the reduction of A , which is the restriction of C_{ε}).

So we have the following situation.

\mathfrak{g}_{rs}	\mathfrak{g}	\mathcal{N}
c_{rs}	c	a
c_{rs}	b	b
	c	a

And we check that we can get $\mathcal{K}_{\mathcal{N}}$ either by Fourier-Deligne transform, or by restriction. For the Springer correspondence, b is missing. It appears neither in the top nor in the socle. Thus it is said to be cuspidal. For the modular generalized Springer correspondence in type A , we refer to [AHJR13] (general linear group) and [AHJR14a] (special linear groups).

Another way to involve b would be to replace the regular representation of the symmetric group by the direct sum of all induced modules from parabolic subgroups (whose endomorphism algebra is the Schur algebra). What we lack here is the induction from \mathfrak{S}_2 to \mathfrak{S}_2 , which gives the trivial module, and hence the constant shifted local system c_{rs} . If we start from the direct sum of this constant local system with \mathcal{K}_{rs} , then taking the intermediate extension and restricting to the nilpotent cone, we get:

\mathfrak{g}_{rs}	\mathfrak{g}	\mathcal{N}
$c_{\text{rs}} \oplus c_{\text{rs}}$	$c \oplus b$	$b \oplus a$
	c	a

where both a and b appear as the top constituents of the restriction. The structure of the direct summands is exactly similar to that of the projective modules of the Schur algebra of GL_2 in degree 2. This is not surprising by the results of [Mau12].

9. TABLES FOR EXCEPTIONAL GROUPS

In this Section we will determine explicitly the modular Springer correspondence for all exceptional groups. Since we consider the non-generalized modular Springer correspondence, only local systems with trivial central character appear, and we may assume that G is simple and adjoint.

Theorem 5.2 is a powerful tool to determine the modular Springer correspondence, if one knows enough about the decomposition matrix of the Weyl group: the case of GL_n was treated this way in Theorem 7.1. In fact, the results in Subsection 5.3 imply that we can determine explicitly the modular Springer correspondence for any given type if the decomposition matrix of the Weyl group is known. Much better, the knowledge of the ordinary character table of the Weyl group is actually enough: consider the matrix of the restrictions of the characters to the ℓ -regular conjugacy classes, in some total order compatible with the Springer correspondence (nilpotent orbits are ordered by closure inclusions, and we choose a suitable order on the characters of each component group). Then the Springer basic set consists of the characters whose restriction is not in the linear span of the preceding restrictions (i.e. the lines making the rank increase by 1, from top to bottom). Let us give the example of G_2 for $\ell = 2$ to illustrate this:

0	$\chi_{1,0}$	1	1
A_1	$\chi_{1,3}$	1	1
\bar{A}_1	$\chi_{2,2}$	2	-1
$G_2(a_1), 3$	$\chi_{2,1}$	2	-1
$G_2(a_1), 21$	$\chi'_{1,3}$	1	1
G_2	$\chi_{1,6}$	1	1

The notation for the characters of the Weyl group is the standard one: the first integer denotes the degree, and the second one denotes the first symmetric power of the natural representation where the character appears. The notation for pairs is as follows: if the component group is trivial, we only write the orbit; otherwise it is a symmetric group, and we indicate the irreducible local system by a partition. Of course the trivial character is in the image of the Springer correspondence. The next character $\chi'_{1,3}$ has the same restriction as the trivial character, hence it is not in the Springer basic set. So the orbit A_1 (with the trivial local system) is not in the image of the modular Springer correspondence. However the character $\chi_{2,2}$ has a restriction which is not in the span of the preceding lines, hence it is in the Springer basic set. The modular reduction is the sum of a new Brauer character and possibly some copies of the trivial one (actually none, since they are not in the same block; but for the sake of explaining this method, let us pretend that we do not know it). Despite this ambiguity, this is already enough information to characterize this new Brauer character (although we do not know it explicitly), and to know that it is sent to \tilde{A}_1 via the modular Springer correspondence.

We could have proceeded that way, however for Weyl groups of exceptional types, the decomposition matrices are completely known anyway (see [Kho84, KM85] for $\ell > 2$, and they are obtainable by GAP3 in all exceptional types for all characteristics), so we will display the information in the form of those decomposition matrices. The image of the modular Springer correspondence will be given by the top 1's in each column. As usual, it is more convenient to display the information block by block. Blocks of defect zero behave just as in characteristic zero. For blocks of defect one, one can display the information in the form of a Brauer tree (which here is a line, because all characters are rational, hence real; besides, there is no exceptional vertex). For each block of defect one, there is one pair not appearing in the modular Springer correspondence. We felt that, even if the decomposition matrices were already known, it was useful to include them for several reasons: we display them all together, blockwise, with the characters ordered via the Springer correspondence, with labels both in terms of characters and in terms of orbits and local systems, with a uniform and standard notation. In this way, the ordinary and modular Springer correspondences can be read off easily from them.

9.1. Type G_2 .

9.1.1. *Case $\ell = 2$.* We have the principal block, of defect two:

$$\begin{array}{cc|c} 0 & \chi_{1,0} & 1 \\ A_1 & \chi'_{1,3} & 1 \\ G_2(a_1), 21 & \chi''_{1,3} & 1 \\ G_2 & \chi_{1,6} & 1 \end{array}$$

and another block of defect one:

$$\begin{array}{cc|c} \tilde{A}_1 & \chi_{2,2} & 1 \\ G_2(a_1), 3 & \chi_{2,1} & 1 \end{array}$$

This confirms that the Springer basic set is $\{\chi_{1,0}, \chi_{2,2}\}$, and that the image of the modular Springer correspondence is $\{0, \tilde{A}_1\}$.

9.1.2. *Case $\ell = 3$.* We have two blocks of defect one:

$$\begin{array}{cc|cc} 0 & \chi_{1,0} & 1 & \cdot \\ \tilde{A}_1 & \chi_{2,2} & 1 & 1 \\ G_2 & \chi_{1,6} & \cdot & 1 \end{array} \quad \text{and} \quad \begin{array}{cc|cc} A_1 & \chi'_{1,3} & 1 & \cdot \\ G_2(a_1), 3 & \chi_{2,1} & 1 & 1 \\ G_2(a_1), 21 & \chi''_{1,3} & \cdot & 1 \end{array}$$

This could be summarized by the Brauer trees:

$$\chi_{1,0} \text{ --- } \chi_{2,2} \text{ --- } \chi_{1,6} \quad \text{and} \quad \chi'_{1,3} \text{ --- } \chi_{2,1} \text{ --- } \chi''_{1,3}$$

The corresponding pairs are:

$$0 \text{ --- } \tilde{A}_1 \text{ --- } G_2 \quad \text{and} \quad A_1 \text{ --- } (G_2(a_1), 3) \text{ --- } (G_2(a_1), 21)$$

The Springer basic set is $\{\chi_{1,0}, \chi_{2,2}, \chi'_{1,3}, \chi_{2,1}\}$. The associated irreducible Brauer characters correspond to the pairs 0 , \tilde{A}_1 , A_1 and $(G_2(a_1), 3)$. The missing pairs are at the end of the trees (on the side of the largest nilpotent orbit). Note that the modular pair $(G_2(a_1), 21)$ is the modular reduction of the cuspidal pair from characteristic zero coefficients, hence we already knew it should not be involved in the modular Springer correspondence.

9.2. Type F_4 .

9.2.1. *Case $\ell = 2$.* We only have the principal block. The image of the modular Springer correspondence is $\{0, A_1, (\tilde{A}_1, 2), A_1 + \tilde{A}_1\}$.

0	$\chi_{1,0}$	1	.	.	.
A_1	$\chi_{2,4}$.	1	.	.
$\tilde{A}_1, 2$	$\chi_{4,1}$.	1	1	.
$\tilde{A}_1, 11$	$\chi''_{2,4}$.	.	1	.
$A_1 + \tilde{A}_1$	$\chi_{9,2}$	1	1	1	1
$A_2, 2$	$\chi_{8,3}$	2	1	.	1
$A_2, 11$	$\chi'_{1,12}$	1	.	.	.
\tilde{A}_2	$\chi_{8,3}$	2	.	1	1
$A_2 + \tilde{A}_1$	$\chi_{4,7}$.	1	1	.
$B_2, 2$	$\chi_{9,6}$	1	1	1	1
$\tilde{A}_2 + A_1$	$\chi_{6,6}$	2	.	.	1
$B_2, 11$	$\chi_{4,8}$.	.	.	1
$C_3(a_1), 2$	$\chi_{16,5}$.	2	2	2
$C_3(a_1), 11$	$\chi'_{4,7}$.	1	1	.
$F_4(a_3), 4$	$\chi_{12,4}$.	2	2	1
$F_4(a_3), 31$	$\chi_{9,6}$	1	1	1	1
$F_4(a_3), 22$	$\chi_{6,6}$	2	.	.	1
$F_4(a_3), 211$	$\chi'_{1,12}$	1	.	.	.
C_3	$\chi_{8,9}$	2	1	.	1
B_3	$\chi_{8,9}$	2	.	1	1
$F_4(a_2), 2$	$\chi_{9,10}$	1	1	1	1
$F_4(a_2), 11$	$\chi_{2,16}$.	.	1	.
$F_4(a_1), 2$	$\chi_{4,13}$.	1	1	.
$F_4(a_1), 11$	$\chi'_{2,16}$.	1	.	.
F_4	$\chi_{1,24}$	1	.	.	.

9.2.2. *Case $\ell = 3$.* We have two blocks of defect two:

0	$\chi_{1,0}$	1	.	.	.
A_1	$\chi_{2,4}$	1	1	.	.
$\tilde{A}_1, 11$	$\chi''_{2,4}$	1	.	1	.
$A_2, 11$	$\chi'_{1,12}$.	1	.	.
$B_2, 11$	$\chi_{4,8}$	1	1	1	1
$F_4(a_3), 211$	$\chi'_{1,12}$.	.	1	.
$F_4(a_2), 11$	$\chi'_{2,16}$.	1	.	1
$F_4(a_1), 11$	$\chi_{2,16}$.	.	1	1
F_4	$\chi_{1,24}$.	.	.	1

and

$\tilde{A}_1, 2$	$\chi_{4,1}$	1	.	.	.
$A_2, 2$	$\chi_{8,3}$	1	1	.	.
\tilde{A}_2	$\chi_{8,3}$	1	.	1	.
$A_2 + \tilde{A}_1$	$\chi_{4,7}$.	1	.	.
$C_3(a_1), 2$	$\chi_{16,5}$	1	1	1	1
$C_3(a_1), 11$	$\chi'_{4,7}$.	.	1	.
C_3	$\chi_{8,9}$.	.	1	1
B_3	$\chi_{8,9}$.	1	.	1
$F_4(a_1), 2$	$\chi_{4,13}$.	.	.	1

There is one block of defect one, described by the following tree:

$$\chi''_{6,6} \text{ --- } \chi_{12,4} \text{ --- } \chi'_{6,6}$$

corresponding to the following pairs:

$$A_2 + A_1 \text{ --- } (F_4(a_3), 4) \text{ --- } (F_4(a_3), 22)$$

Finally, the other characters each form a block of defect zero:

$$\begin{array}{cccc} \chi_{9,2} & \chi'_{9,6} & \chi''_{9,6} & \chi_{9,10} \\ A_1 + A_1 & (B_2, 2) & (F_4(a_3), 31) & (F_4(a_2), 2) \end{array}$$

9.3. Type E_6 .

9.3.1. *Case $\ell = 2$.* We have the principal, block of defect 7:

0	$\chi_{1,0}$	1
A_1	$\chi_{6,1}$.	1	.	.	.
$2A_1$	$\chi_{20,2}$.	1	1	.	.
$3A_1$	$\chi_{15,4}$	1	1	.	1	.
$A_2, 2$	$\chi_{30,3}$	2	1	1	1	.
$A_2, 11$	$\chi_{15,5}$	1	.	1	.	.
$2A_2$	$\chi_{24,6}$	2	.	1	1	.
$A_2 + 2A_1$	$\chi_{60,5}$.	1	1	.	1
A_3	$\chi_{81,6}$	1	3	1	1	1
$2A_2 + A_1$	$\chi_{10,9}$	2	.	.	1	.
$A_3 + A_1$	$\chi_{60,8}$.	2	.	1	1
$D_4(a_1), 3$	$\chi_{80,7}$.	4	.	2	1
$D_4(a_1), 21$	$\chi_{90,8}$	2	2	2	1	1
$D_4(a_1), 111$	$\chi_{20,10}$.	2	.	1	.
D_4	$\chi_{24,12}$	2	.	1	1	.
A_4	$\chi_{81,10}$	1	3	1	1	1
$A_4 + A_1$	$\chi_{60,11}$.	1	1	.	1
A_5	$\chi_{15,16}$	1	1	.	1	.
$E_6(a_3), 2$	$\chi_{30,15}$	2	1	1	1	.
$E_6(a_3), 11$	$\chi_{15,17}$	1	.	1	.	.
D_5	$\chi_{20,20}$.	1	1	.	.
$E_6(a_1)$	$\chi_{6,25}$.	1	.	.	.
E_6	$\chi_{1,36}$	1

and a block of defect one:

$$\begin{array}{cc|c} A_2 + A_1 & \chi_{64,4} & 1 \\ D_5(a_1) & \chi_{64,13} & 1 \end{array}$$

9.3.2. *Case $\ell = 3$.* We have the principal block, of defect four:

0	$\chi_{1,0}$	1
A_1	$\chi_{6,1}$	1	1
$2A_1$	$\chi_{20,2}$	1	1	1
$3A_1$	$\chi_{15,4}$.	.	1	1
$A_2, 2$	$\chi_{30,3}$.	1	.	.	1
$A_2, 11$	$\chi_{15,5}$.	1	.	.	.	1	.	.	.
$A_2 + A_1$	$\chi_{64,4}$.	1	1	.	1	1	1	.	.
$2A_2$	$\chi_{24,6}$.	.	1	.	.	.	1	.	.
$A_2 + 2A_1$	$\chi_{60,5}$.	1	1	1	1	.	1	1	.
$2A_2 + A_1$	$\chi_{10,9}$.	1	1	.
$A_3 + A_1$	$\chi_{60,8}$	1	1	1	1	.	1	1	1	1
$D_4(a_1), 3$	$\chi_{80,7}$.	1	.	.	1	1	1	1	.
$D_4(a_1), 21$	$\chi_{90,8}$	1	2	2	.	1
$D_4(a_1), 111$	$\chi_{20,10}$	1	1	.	.
D_4	$\chi_{24,12}$	1	.	1	.
$A_4 + A_1$	$\chi_{60,11}$	1	1	.	.	.	1	.	1	1
A_5	$\chi_{15,16}$	1	1
$D_5(a_1)$	$\chi_{64,13}$	1	1	1	1
$E_6(a_3), 2$	$\chi_{30,15}$	1	.
$E_6(a_3), 11$	$\chi_{15,17}$	1	1	.
D_5	$\chi_{20,20}$.	.	.	1	.	.	.	1	1
$E_6(a_1)$	$\chi_{6,25}$.	.	.	1	.	.	.	1	.
E_6	$\chi_{1,36}$.	.	.	1

and two blocks of defect zero:

$$\begin{array}{cc|c} A_3 & \chi_{81,6} & 1 \end{array} \quad \text{and} \quad \begin{array}{cc|c} A_4 & \chi_{81,10} & 1 \end{array}$$

$$\begin{array}{cc|ccc}
0 & \chi_{1,0} & 1 & . & . & . \\
2A_2 & \chi_{24,6} & 1 & 1 & . & . \\
A_4 & \chi_{81,10} & . & 1 & 1 & . \\
D_5(a_1) & \chi_{64,13} & . & . & 1 & 1 \\
E_6(a_1) & \chi_{6,25} & . & . & . & 1
\end{array}
\quad \text{and} \quad
\begin{array}{cc|ccc}
A_1 & \chi_{6,1} & 1 & . & . & . \\
A_2+A_1 & \chi_{64,4} & 1 & 1 & . & . \\
A_3 & \chi_{81,6} & . & 1 & 1 & . \\
D_4 & \chi_{24,12} & . & . & 1 & 1 \\
E_6 & \chi_{1,36} & . & . & . & 1
\end{array}$$

$X_{20,2}$	$2A_1$	$X_{10,9}$	$2A_2+A_1$	$X_{60,11}$	A_4+A_1
$X_{15,4}$	$3A_1$	$X_{60,8}$	A_3+A_1	$X_{15,16}$	A_5
$X_{30,3}$	$(A_2, 2)$	$X_{80,7}$	$(D_4(a_1), 3)$	$X_{30,15}$	$(E_6(a_3), 2)$
$X_{15,5}$	$(A_2, 11)$	$X_{90,8}$	$(D_4(a_1), 21)$	$X_{15,17}$	$(E_6(a_3), 11)$
$X_{60,5}$	A_2+2A_1	$X_{20,10}$	$(D_4(a_1), 111)$	$X_{20,20}$	D_5

9.4.1. *Case $\ell = 2$.* We have the principal block, of defect 10:

0	X1,0	1	1
A ₁	X7,1	1	1	1
2A ₁ '	X27,2	1	2	1
3A ₁ '	X21,3	1	1	1	1
3A ₁ '	X35,4	1	2	1	1
A ₂ , 2	X56,3	2	3	2	1
A ₂ , 11	X21,6	1	1	1	1
4A ₁	X15,7	1	1	.	1
A ₂ +A ₁ , 2	X120,4	2	3	2	1	1	.	.	.
A ₂ +A ₁ , 11	X105,5	1	2	2	.	1	.	.	.
A ₂ +2A ₁	X189,5	3	4	3	1	1	1	1	1
A ₃	X210,6	4	5	4	1	1	1	1	1
A ₂ +3A ₁	X105,6	3	3	2	1	.	1	1	1
2A ₂	X168,6	2	2	3	.	1	1	1	1
(A ₃ +A ₁)''	X189,7	3	4	3	1	1	1	1	1
2A ₂ +A ₁	X70,9	2	2	1	1	.	1	1	1
(A ₃ +A ₁)'	X280,8	2	3	2	1	1	1	1	1
D ₄ (a ₁), 3	X315,7	3	5	3	2	1	1	1	1
D ₄ (a ₁), 111	X35,13	2	2	1	1
A ₃ +2A ₁	X216,9	2	3	2	1	.	1	1	1
D ₄ (a ₁), 21	X280,9	6	6	5	1	1	2	.	.
D ₄ (a ₁)+A ₁ , 11	X189,10	3	4	3	1	1	1	1	1
D ₄	X105,12	3	3	2	1	.	1	1	1
D ₄ (a ₁)+A ₁ , 2	X405,8	5	7	5	2	1	2	1	1
A ₃ +A ₂ , 2	X378,9	4	5	4	2	1	2	1	1
A ₃ +A ₂ , 11	X84,12	2	2	1	1	.	1	1	1
A ₄ , 11	X336,11	4	6	4	2	1	1	1	1
A ₃ +A ₂ +A ₁	X210,10	2	3	1	2	.	1	1	1
A ₄ , 2	X420,10	6	8	5	3	1	2	1	1
D ₄ +A ₁	X84,15	2	2	1	1	.	1	1	1
A ₅ '	X105,15	3	3	2	1	.	1	1	1
D ₅ (a ₁), 2	X420,13	6	8	5	3	1	2	1	1
D ₅ (a ₁), 11	X336,14	4	6	4	2	1	1	1	1
A ₄ +A ₂	X210,13	2	3	1	2	.	1	1	1
A ₅ +A ₁	X70,18	2	1	1	.	1	1	1	1
D ₅ (a ₁)+A ₁	X378,14	4	5	4	2	1	2	1	1
A ₅	X216,16	2	3	2	1	.	1	1	1
D ₆ (a ₂)	X280,17	2	3	2	1	1	1	1	1
E ₆ (a ₃), 2	X405,15	5	7	5	2	1	2	1	1
E ₆ (a ₃), 11	X189,17	3	4	3	1	1	1	1	1
E ₇ (a ₅), 3	X315,16	3	5	3	2	1	1	1	1
E ₇ (a ₅), 21	X280,18	6	6	5	1	1	2	.	.
E ₇ (a ₅), 111	X35,22	1	2	1	1
D ₅	X189,20	3	4	3	1	1	1	1	1
A ₆	X105,21	3	3	2	1	.	1	1	1
D ₆ (a ₁)	X210,21	4	5	4	1	1	1	1	1
D ₅ +A ₁	X168,21	2	2	3	.	1	1	1	1
E ₇ (a ₄), 2	X189,22	3	4	3	1	1	1	1	1
E ₇ (a ₄), 11	X15,28	1	1	.	1
E ₆ (a ₁), 2	X120,25	2	3	2	1	1	.	.	.
D ₆	X35,31	1	2	1	1
E ₆ (a ₁), 11	X105,26	1	2	2	.	1	.	.	.
E ₆	X21,36	1	1	1	.	1	.	.	.
E ₇ (a ₃), 2	X56,30	2	3	2	1
E ₇ (a ₃), 11	X21,33	1	1	1
E ₇ (a ₂)	X27,37	1	2	1
E ₇ (a ₁)	X7,46	1	1
E ₇	X1,63	1

and a block of defect one:

$$\begin{array}{cc|c} A_4+A_1, 2 & \chi_{512,12} & 1 \\ A_4+A_1, 11 & \chi_{512,11} & 1 \end{array}$$

9.4.2. *Case $\ell = 3$.* We have two blocks of defect four:

$$\begin{array}{cc|cccccccccccc} 0 & \chi_{1,0} & 1 & . & . & . & . & . & . & . & . & . & . \\ 3A_1' & \chi_{35,4} & 1 & 1 & . & . & . & . & . & . & . & . & . \\ A_2, 11 & \chi_{21,6} & . & . & 1 & . & . & . & . & . & . & . & . \\ A_2+A_1, 2 & \chi_{120,4} & 1 & . & 1 & 1 & . & . & . & . & . & . & . \\ A_3 & \chi_{210,6} & . & . & 1 & 1 & 1 & . & . & . & . & . & . \\ A_2+3A_1 & \chi_{105,6} & . & . & . & 1 & . & 1 & . & . & . & . & . \\ 2A_2 & \chi_{168,6} & 1 & 1 & . & 1 & . & . & 1 & . & . & . & . \\ (A_3+A_1)' & \chi_{280,8} & 1 & 1 & . & 1 & 1 & 1 & 1 & 1 & . & . & . \\ D_4 & \chi_{105,12} & . & . & . & . & 1 & . & . & 1 & . & . & . \\ A_3+A_2, 11 & \chi_{84,12} & 1 & 1 & . & . & . & . & 1 & 1 & . & . & . \\ A_3+A_2+A_1 & \chi_{210,10} & . & . & 1 & 1 & . & 1 & 1 & . & 1 & . & . \\ A_4, 2 & \chi_{420,10} & . & . & . & 1 & 1 & . & 1 & . & . & . & 1 \\ A_4+A_1, 2 & \chi_{512,12} & 1 & . & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ D_5(a_1), 11 & \chi_{336,14} & . & . & . & . & 1 & . & 1 & 1 & . & . & 1 \\ A_5+A_1 & \chi_{70,18} & . & . & 1 & . & . & . & . & . & 1 & . & . \\ E_7(a_5), 3 & \chi_{315,16} & . & . & 1 & . & . & . & . & . & 2 & 1 & . \\ E_7(a_5), 21 & \chi_{280,18} & . & . & . & . & . & . & 1 & . & . & 1 & 1 \\ E_7(a_5), 111 & \chi_{35,22} & . & . & . & . & . & . & 1 & . & . & . & . \\ E_7(a_4), 11 & \chi_{15,28} & 1 & . & . & . & . & . & . & 1 & . & . & . \\ E_6(a_1), 11 & \chi_{105,26} & . & . & . & . & . & 1 & 1 & 1 & 1 & . & . \\ E_6 & \chi_{21,36} & . & . & . & . & . & 1 & . & 1 & . & . & . \\ E_7(a_3), 2 & \chi_{56,30} & . & . & . & . & . & 1 & . & . & 1 & . & . \\ E_7(a_1) & \chi_{7,46} & . & . & . & . & . & 1 & . & . & . & . & . \end{array}$$

and

$$\begin{array}{cc|cccccccccccc} A_1 & \chi_{7,1} & 1 & . & . & . & . & . & . & . & . & . & . \\ 3A_1'' & \chi_{21,3} & 1 & 1 & . & . & . & . & . & . & . & . & . \\ A_2, 2 & \chi_{56,3} & 1 & . & 1 & . & . & . & . & . & . & . & . \\ 4A_1 & \chi_{15,7} & . & 1 & . & 1 & . & . & . & . & . & . & . \\ A_2+A_1, 11 & \chi_{105,5} & 1 & 1 & 1 & . & 1 & . & . & . & . & . & . \\ 2A_2+A_1 & \chi_{70,9} & . & . & 1 & . & . & 1 & . & . & . & . & . \\ D_4(a_1), 3 & \chi_{315,7} & . & . & 2 & . & . & 1 & 1 & . & . & . & . \\ D_4(a_1), 111 & \chi_{35,13} & . & . & . & . & 1 & . & . & . & . & . & . \\ D_4(a_1), 21 & \chi_{280,9} & . & . & 1 & . & 1 & . & 1 & . & . & . & . \\ A_4, 11 & \chi_{336,11} & . & 1 & . & . & 1 & . & 1 & 1 & . & . & . \\ D_4+A_1 & \chi_{84,15} & . & 1 & . & 1 & 1 & . & . & . & 1 & . & . \\ A_5 & \chi_{105,15} & . & 1 & . & . & . & . & . & 1 & . & . & . \\ A_4+A_1, 11 & \chi_{512,11} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & 1 \\ D_5(a_1), 2 & \chi_{420,13} & . & . & . & . & 1 & . & 1 & 1 & . & . & 1 \\ A_4+A_2 & \chi_{210,13} & 1 & . & 1 & . & 1 & 1 & . & . & . & . & 1 \\ D_6(a_2) & \chi_{280,17} & 1 & 1 & . & 1 & 1 & . & . & 1 & 1 & 1 & 1 \\ A_6 & \chi_{105,21} & 1 & . & . & . & . & . & . & . & . & . & 1 \\ D_6(a_1) & \chi_{210,21} & . & . & . & . & . & 1 & . & 1 & . & . & 1 \\ D_5+A_1 & \chi_{168,21} & . & . & . & 1 & 1 & . & . & . & 1 & 1 & . \\ E_6(a_1), 2 & \chi_{120,25} & . & . & . & 1 & . & 1 & . & . & . & . & 1 \\ D_6 & \chi_{35,31} & . & . & . & 1 & . & . & . & . & 1 & . & . \\ E_7(a_3), 11 & \chi_{21,33} & . & . & . & . & . & 1 & . & . & . & . & . \\ E_7 & \chi_{1,63} & . & . & . & 1 & . & . & . & . & . & . & . \end{array}$$

four blocks of defect one:

$$\begin{array}{cc|c} 2A_1 & \chi_{27,2} & 1 \\ A_5' & \chi_{216,16} & 1 \\ D_5 & \chi_{189,20} & 1 \end{array}$$

$$\begin{array}{cc|c} (A_3+A_1)'' & \chi_{189,7} & 1 \\ A_3+2A_1 & \chi_{216,9} & 1 \\ E_7(a_2) & \chi_{27,37} & 1 \end{array}$$

$$\begin{array}{cc|c} D_4(a_1)+A_1, 11 & \chi_{189,10} & 1 \\ D_5(a_1)+A_1 & \chi_{378,14} & 1 \\ E_7(a_4), 2 & \chi_{189,22} & 1 \end{array}$$

$$\begin{array}{cc|c} A_2+2A_1 & \chi_{189,5} & 1 \\ A_3+A_2, 2 & \chi_{378,9} & 1 \\ E_6(a_3), 11 & \chi_{189,17} & 1 \end{array}$$

and two blocks of defect zero:

$$D_4(a_1)+A_1, 2 \quad \chi_{405,8} \quad | \quad 1 \quad \text{and} \quad E_6(a_3), 2 \quad \chi_{405,15} \quad | \quad 1$$

9.4.3. *Case $\ell = 5$.* There are 6 blocks of defect one, with following Brauer trees:

$$\begin{array}{ccccccccc}
\chi_{1,0} & \text{---} & \chi_{84,12} & \text{---} & \chi_{216,16} & \text{---} & \chi_{189,22} & \text{---} & \chi_{56,30} \\
\chi_{56,3} & \text{---} & \chi_{189,5} & \text{---} & \chi_{216,9} & \text{---} & \chi_{84,15} & \text{---} & \chi_{1,63} \\
\chi_{27,2} & \text{---} & \chi_{168,6} & \text{---} & \chi_{512,12} & \text{---} & \chi_{378,14} & \text{---} & \chi_{7,46} \\
\chi_{7,1} & \text{---} & \chi_{378,9} & \text{---} & \chi_{512,11} & \text{---} & \chi_{168,21} & \text{---} & \chi_{27,37} \\
\chi_{21,6} & \text{---} & \chi_{189,10} & \text{---} & \chi_{336,14} & \text{---} & \chi_{189,20} & \text{---} & \chi_{21,36} \\
\chi_{21,3} & \text{---} & \chi_{189,7} & \text{---} & \chi_{336,11} & \text{---} & \chi_{189,17} & \text{---} & \chi_{21,33}
\end{array}$$

and the corresponding pairs are:

$$\begin{array}{l}
0 \text{ --- } (A_3 + A_2, 11) \text{ --- } A'_5 \text{ --- } (E_7(a_4), 2) \text{ --- } (E_7(a_3), 2) \\
(A_2, 2) \text{ --- } A_2 + 2A_1 \text{ --- } A_3 + 2A_1 \text{ --- } D_4 + A_1 \text{ --- } E_7 \\
2A_1 \text{ --- } 2A_2 \text{ --- } (A_4 + A_1, 2) \text{ --- } D_5(a_1) + A_1 \text{ --- } E_7(a_1) \\
A_1 \text{ --- } (A_3 + A_2, 2) \text{ --- } (A_4 + A_1, 11) \text{ --- } D_5 + A_1 \text{ --- } E_7(a_2) \\
(A_2, 11) \text{ --- } (D_4(a_1) + A_1, 11) \text{ --- } (D_5(a_1), 11) \text{ --- } D_5 \text{ --- } E_6 \\
3A'_1 \text{ --- } (A_3 + A_1)'' \text{ --- } (A_4, 11) \text{ --- } (E_6(a_3), 11) \text{ --- } (E_7(a_3), 11)
\end{array}$$

Finally, there are 30 blocks of defect zero:

$\chi_{35,4}$	$3A'_1$	$\chi_{280,9}$	$(D_4(a_1), 21)$	$\chi_{405,15}$	$(E_6(a_3), 2)$
$\chi_{15,7}$	$4A_1$	$\chi_{105,12}$	D_4	$\chi_{315,16}$	$(E_7(a_5), 3)$
$\chi_{120,4}$	$(A_2 + A_1, 2)$	$\chi_{405,8}$	$(D_4(a_1) + A_1, 2)$	$\chi_{280,18}$	$(E_7(a_5), 21)$
$\chi_{105,5}$	$(A_2 + A_1, 11)$	$\chi_{210,10}$	$A_3 + A_2 + A_1$	$\chi_{35,22}$	$(E_7(a_5), 111)$
$\chi_{210,6}$	A_3	$\chi_{420,10}$	$(A_4, 2)$	$\chi_{105,21}$	A_6
$\chi_{105,6}$	$A_2 + 3A_1$	$\chi_{105,15}$	A'_5	$\chi_{210,21}$	$D_6(a_1)$
$\chi_{70,9}$	$2A_2 + A_1$	$\chi_{420,13}$	$(D_5(a_1), 2)$	$\chi_{15,28}$	$(E_7(a_4), 11)$
$\chi_{280,8}$	$(A_3 + A_1)'$	$\chi_{210,13}$	$A_4 + A_2$	$\chi_{120,25}$	$(E_6(a_1), 2)$
$\chi_{315,7}$	$(D_4(a_1), 3)$	$\chi_{70,18}$	$A_5 + A_1$	$\chi_{35,31}$	D_6
$\chi_{35,13}$	$(D_4(a_1), 111)$	$\chi_{280,17}$	$D_6(a_2)$	$\chi_{105,26}$	$(E_6(a_1), 11)$

So the following pairs are missing:

$$E_7, E_7(a_1), E_7(a_2), (E_7(a_3), 11), (E_7(a_3), 2) \text{ and } E_6.$$

9.4.4. *Case $\ell = 7$.* There are two blocks of defect one:

$$\begin{array}{ccccccccc}
\chi_{1,0} & \text{---} & \chi_{27,2} & \text{---} & \chi_{120,4} & \text{---} & \chi_{405,8} & \text{---} & \chi_{512,12} & \text{---} & \chi_{216,16} & \text{---} & \chi_{15,28} \\
\chi_{15,7} & \text{---} & \chi_{216,9} & \text{---} & \chi_{512,11} & \text{---} & \chi_{405,15} & \text{---} & \chi_{120,25} & \text{---} & \chi_{27,37} & \text{---} & \chi_{1,63}
\end{array}$$

and the corresponding pairs are:

$$\begin{array}{l}
0 \text{ --- } 2A_1 \text{ --- } (A_2 + A_1, 2) \text{ --- } (D_4(a_1) + A_1, 2) \text{ --- } (A_4 + A_1, 2) \text{ --- } A'_5 \text{ --- } (E_7(a_4), 11) \\
4A_1 \text{ --- } A_3 + 2A_1 \text{ --- } (A_4 + A_1, 11) \text{ --- } (E_6(a_3), 2) \text{ --- } (E_6(a_1), 2) \text{ --- } E_7(a_2) \text{ --- } E_7
\end{array}$$

There are 46 blocks of defect zero:

$\chi_{7,1}$	A_1	$\chi_{189,10}$	$(D_4(a_1) + A_1, 11)$	$\chi_{315,16}$	$(E_7(a_5), 3)$
$\chi_{21,3}$	$3A'_1$	$\chi_{105,12}$	D_4	$\chi_{280,18}$	$(E_7(a_5), 21)$
$\chi_{35,4}$	$3A'_1$	$\chi_{378,9}$	$(A_3 + A_2, 2)$	$\chi_{35,22}$	$(E_7(a_5), 111)$
$\chi_{56,3}$	$(A_2, 2)$	$\chi_{84,12}$	$(A_3 + A_2, 11)$	$\chi_{189,20}$	D_5
$\chi_{21,6}$	$(A_2, 11)$	$\chi_{336,11}$	$(A_4, 11)$	$\chi_{105,21}$	A_6
$\chi_{105,5}$	$(A_2 + A_1, 11)$	$\chi_{210,10}$	$A_3 + A_2 + A_1$	$\chi_{210,21}$	$D_6(a_1)$
$\chi_{189,5}$	$A_2 + 2A_1$	$\chi_{420,10}$	$(A_4, 2)$	$\chi_{168,21}$	$D_5 + A_1$
$\chi_{210,6}$	A_3	$\chi_{84,15}$	$D_4 + A_1$	$\chi_{189,22}$	$(E_7(a_4), 2)$
$\chi_{105,6}$	$A_2 + 3A_1$	$\chi_{105,15}$	A'_5	$\chi_{35,31}$	D_6
$\chi_{168,6}$	$2A_2$	$\chi_{420,13}$	$(D_5(a_1), 2)$	$\chi_{105,26}$	$(E_6(a_1), 11)$
$\chi_{189,7}$	$(A_3 + A_1)''$	$\chi_{336,14}$	$(D_5(a_1), 11)$	$\chi_{21,36}$	E_6
$\chi_{70,9}$	$2A_2 + A_1$	$\chi_{210,13}$	$A_4 + A_2$	$\chi_{56,30}$	$(E_7(a_3), 2)$
$\chi_{280,8}$	$(A_3 + A_1)'$	$\chi_{70,18}$	$A_5 + A_1$	$\chi_{21,33}$	$(E_7(a_3), 11)$
$\chi_{315,7}$	$(D_4(a_1), 3)$	$\chi_{378,14}$	$D_5(a_1) + A_1$	$\chi_{7,46}$	$E_7(a_1)$
$\chi_{35,13}$	$(D_4(a_1), 111)$	$\chi_{280,17}$	$D_6(a_2)$		
$\chi_{280,9}$	$(D_4(a_1), 21)$	$\chi_{189,17}$	$(E_6(a_3), 11)$		

The pairs E_7 and $(E_7(a_4), 11)$ are missing.

9.5. Type E_8 .

9.5.1. *Case $\ell = 2$.* The principal block has defect 14. To save space, we put on the same line a character and its twist by the sign character. The left column shows the character of the pair which appears first in the Springer order.

[illegible]

The other block has defect two:

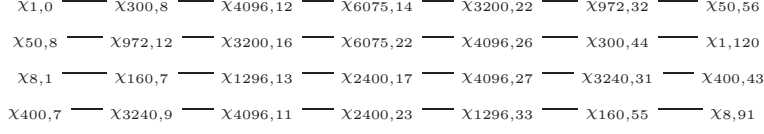
$A_4+A_1, 2$	$\chi_{4096,12}$	1
$A_4+A_1, 11$	$\chi_{4096,11}$	1
$E_6(a_1)+A_1, 2$	$\chi_{4096,27}$	1
$E_6(a_1)+A_1, 11$	$\chi_{4096,26}$	1

So the image of the 2-modular Springer correspondence is:

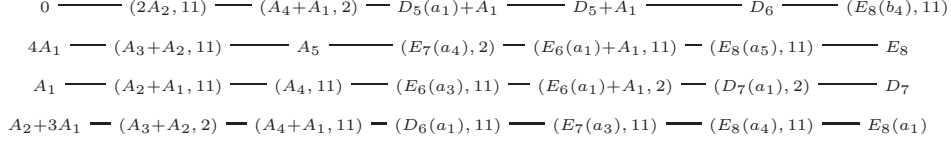
$$\{0, A_1, 2A_1, 3A_1, 4A_1, (A_2 + A_1, 2), A_2 + 2A_1, A_2 + 3A_1, A_3 + A_1, A_3 + 2A_1, (A_4 + A_1, 2), (A_4 + 2A_1, 2)\}.$$

[illegible]
$$\begin{array}{llll}
\chi_{567,6} & \text{---} & \chi_{2835,22} & \text{---} & \chi_{2268,30} & A_3 & \text{---} & A_6+A_1 & \text{---} & (E_7(a_3), 2) \\
\chi_{2268,10} & \text{---} & \chi_{2835,14} & \text{---} & \chi_{567,46} & (A_4, 2) & \text{---} & A_4+A_2+A_1 & \text{---} & E_7(a_1) \\
\chi_{1134,20} & \text{---} & \chi_{5670,18} & \text{---} & \chi_{4536,18} & (E_6(a_3)+A_1, 11) & \text{---} & (E_8(a_7), 41) & \text{---} & (E_8(a_7), 32) \\
\chi_{1296,13} & \text{---} & \chi_{4536,23} & \text{---} & \chi_{3240,31} & (A_4, 11) & \text{---} & (D_5+A_2, 2) & \text{---} & (D_7(a_1), 2) \\
\chi_{3240,9} & \text{---} & \chi_{4536,13} & \text{---} & \chi_{1296,33} & (A_3+A_2, 2) & \text{---} & A_4+A_2 & \text{---} & (E_7(a_3), 11)
\end{array}$$
$$\begin{array}{cccc} \chi_{972,12} & \chi_{6075,14} & \chi_{6075,22} & \chi_{972,32} \\ (A_3+A_2, 11) & D_5(a_1)+A_1 & (E_7(a_4), 2) & D_6 \end{array}$$

9.5.4. *Case $\ell = 7$.* There are four blocks of defect one, with the following Brauer trees:



and corresponding pairs



There are 84 blocks of defect zero:

$\chi_{35,2}$	$2A_1$	$\chi_{3360,13}$	$(A_4 + 2A_1, 11)$	$\chi_{2100,28}$	$(E_6(a_1), 11)$
$\chi_{84,4}$	$3A_1$	$\chi_{4536,13}$	$A_4 + A_2$	$\chi_{4536,23}$	$(D_5 + A_2, 2)$
$\chi_{112,3}$	$(A_2, 2)$	$\chi_{2835,14}$	$A_4 + A_2 + A_1$	$\chi_{840,26}$	$(D_5 + A_2, 11)$
$\chi_{28,8}$	$(A_2, 11)$	$\chi_{4200,15}$	$(D_4 + A_2, 2)$	$\chi_{525,36}$	E_6
$\chi_{210,4}$	$(A_2 + A_1, 2)$	$\chi_{168,24}$	$(D_4 + A_2, 11)$	$\chi_{4200,24}$	$(D_7(a_2), 2)$
$\chi_{560,5}$	$A_2 + 2A_1$	$\chi_{5600,15}$	$(E_6(a_3), 2)$	$\chi_{3360,25}$	$(D_7(a_2), 11)$
$\chi_{567,6}$	A_3	$\chi_{2100,20}$	D_5	$\chi_{1400,29}$	A_7
$\chi_{700,6}$	$(2A_2, 2)$	$\chi_{420,20}$	$A_4 + A_3$	$\chi_{2268,30}$	$(E_7(a_3), 2)$
$\chi_{448,9}$	$2A_2 + A_1$	$\chi_{2016,19}$	$A_5 + A_1$	$\chi_{2240,28}$	$(E_8(b_6), 3)$
$\chi_{1344,8}$	$A_3 + A_1$	$\chi_{1344,19}$	$D_5(a_1) + A_2$	$\chi_{175,36}$	$(E_8(b_6), 21)$
$\chi_{1400,7}$	$(D_4(a_1), 3)$	$\chi_{3150,18}$	$(E_6(a_3) + A_1, 2)$	$\chi_{840,31}$	$(E_8(b_6), 111)$
$\chi_{1008,9}$	$(D_4(a_1), 21)$	$\chi_{1134,20}$	$(E_6(a_3) + A_1, 11)$	$\chi_{448,39}$	$E_6 + A_1$
$\chi_{56,19}$	$(D_4(a_1), 111)$	$\chi_{4200,18}$	$(D_6(a_2), 2)$	$\chi_{1050,34}$	$(D_7(a_1), 11)$
$\chi_{175,12}$	$2A_2 + 2A_1$	$\chi_{2688,20}$	$(D_6(a_2), 11)$	$\chi_{1575,34}$	$(E_8(a_6), 21)$
$\chi_{525,12}$	D_4	$\chi_{7168,17}$	$(E_7(a_5), 3)$	$\chi_{1344,38}$	$E_7(a_2)$
$\chi_{1050,10}$	$A_3 + 2A_1$	$\chi_{5600,19}$	$(E_7(a_5), 21)$	$\chi_{1400,32}$	$(E_8(a_6), 3)$
$\chi_{1400,8}$	$(D_4(a_1) + A_1, 3)$	$\chi_{448,25}$	$(E_7(a_5), 111)$	$\chi_{350,38}$	$(E_8(a_6), 111)$
$\chi_{1575,10}$	$(D_4(a_1) + A_1, 21)$	$\chi_{4480,16}$	$(E_8(a_7), 5)$	$\chi_{1400,37}$	$(E_8(b_5), 3)$
$\chi_{350,14}$	$(D_4(a_1) + A_1, 111)$	$\chi_{5670,18}$	$(E_8(a_7), 41)$	$\chi_{1008,39}$	$(E_8(b_5), 21)$
$\chi_{2268,10}$	$(A_4, 2)$	$\chi_{1680,22}$	$(E_8(a_7), 311)$	$\chi_{56,49}$	$(E_8(b_5), 111)$
$\chi_{1400,11}$	$A_3 + A_2 + A_1$	$\chi_{4536,18}$	$(E_8(a_7), 32)$	$\chi_{700,42}$	$(E_8(a_5), 2)$
$\chi_{2240,10}$	$(D_4(a_1) + A_2, 2)$	$\chi_{1400,20}$	$(E_8(a_7), 221)$	$\chi_{567,46}$	$E_7(a_1)$
$\chi_{840,13}$	$(D_4(a_1) + A_2, 11)$	$\chi_{70,32}$	$(E_8(a_7), 2111)$	$\chi_{560,47}$	$(E_8(b_4), 2)$
$\chi_{700,16}$	$D_4 + A_1$	$\chi_{5600,21}$	$(D_6(a_1), 2)$	$\chi_{84,64}$	E_7
$\chi_{840,14}$	$2A_3$	$\chi_{4200,21}$	A_6	$\chi_{210,52}$	$(E_8(a_4), 2)$
$\chi_{2800,13}$	$(D_5(a_1), 2)$	$\chi_{700,28}$	$(E_7(a_4), 11)$	$\chi_{112,63}$	$(E_8(a_3), 2)$
$\chi_{2100,16}$	$(D_5(a_1), 11)$	$\chi_{2835,22}$	$A_6 + A_1$	$\chi_{28,68}$	$(E_8(a_3), 11)$
$\chi_{4200,12}$	$(A_4 + 2A_1, 2)$	$\chi_{2800,25}$	$(E_6(a_1), 2)$	$\chi_{35,74}$	$E_8(a_2)$

The pairs E_8 , $E_8(a_1)$, $(E_8(b_4), 11)$, D_7 and $(E_8(a_7), 11111)$ are missing, the latter being the modular reduction of the characteristic zero cuspidal pair.

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